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Perturbative Approach to an orbital evolution around a Supermassive black hole

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February 7, 2008

abstract

A charge-free, point particle of infinitesimal mass orbiting a Kerr black hole is known to move along a geodesic. When the particle has a finite mass or charge, it emits radiation which carries away orbital energy and angular momentum, and the orbit deviates from a geodesic.

In this paper we assume that the deviation is small and show that the half-advanced minus half-retarded field surprisingly provides the correct radiation reaction force, in a time-averaged sense, and determines the orbit of the particle.

I. INTRODUCTION

Binary systems of solar-mass compact objects and supermassive black holes at galactic nuclei are expected to be important sources of gravitational waves. In order to detect these gravitational waves, a project to construct a space-based detector, LISA, is underway [1]. The detection of such gravitational waves will reveal fundamental information about gravitational theory, and will provide insights into conditions at the centers of distant galaxies. It may be possible to read out the detailed geometrical structure of a supermassive black hole encoded in the incoming gravitational waves, with which we can test gravitational theory in the strong gravitational region, and our understanding of black holes. Not only for extracting such information from detected gravitational waves, but also for a more efficient detector design of the presently on-going project, it is an urgent problem to calculate as precise as possible the gravitational wave signal expected from such a binary system.

Because of the extreme mass ratio, such a binary system can be treated by a perturbation formalism. We treat the supermassive black hole as a background, and treat the solar-mass compact object as a source of metric perturbation. It was shown† that, when the spatial volumn of the solar-mass compact object is smaller than the background curvature scale, one can approximately use a point source

$$T^{\mu\nu} = \mu \int d\tau \frac{\delta^{(4)}\big(x - z(\tau)\big)}{\sqrt{-g}} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} ,$$

where $z^\mu(\tau)$ is the orbit of the object, and τ is proper time.

By the uniqueness theorem of black holes, we can assume that the background black hole is described by the Kerr geometry. In this case, given an orbit, there is a formalism to calculate the metric perturbation at infinity [3]. Therefore, the remaining problem is how to calculate the orbit $z^\mu(\tau)$ of the source.

In the massless limit, the particle moves along a geodesic. When the particle has a mass, it becomes a source of gravitational radiation, which carries away orbital energy and angular momentum. Thus the orbit deviates from a

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†In the framework of metric perturbations, it is not a simple problem to define a point particle. When we take the zero-volumn limit, the perturbations become divergent around the particle and the perturbation scheme becomes invalid. In Ref. [2], by using a matched asymptotic technique consistent with the perturbation formalism, we showed that the use of a point particle is still valid to induce a correct metric pertrubation.

geodesic. The deviation from a geodesic can be derived fully by solving an equation of motion with the so-called self-force. A general calculation scheme for the gravitational self-force was proposed by Ref. [4]. A similar situation happens to a particle with small scalar/electromagnetic charge, and the scalar and electromagnetic self-forces were proposed in Ref. [5]. In this paper, starting from a discussion of a symmetry property of the self-force, we propose a method to calculate the orbital evolution under scalar/electromagnetic/gravitational radiation reaction. We assume that the effect of the radiation reaction is weak, and consider the leading correction to the orbital evolution.

It is commonly assumed that the orbit evolves in an adiabatic manner, namely, the orbit evolves slowly in its phase space. A true geodesic around a Kerr black hole is characterized by the energy E, the angular momentum L and the Carter constant K[‡]. Numerous investigations have been made to calculate the radiation reaction effect on energy E and angular momentum L [3] by analyzing the asymptotic gravitational waves at infinity and the horizon. However, the calculation of the radiation reaction effect on the Carter constant K is so far an unsolved problem.

A geodesic equation is a set of second order differential equations of 4 functions $\{z^\alpha\}$. With a proper time τ as the parameter characterizing the orbit, we have 7 integral constants, $z^\alpha(\tau=0)$ and $dz^\alpha/d\tau(\tau=0)$, three of which are related to E, L and K. In Sec.II, we introduce a specific symmetry property of families of geodesics in the Kerr geometry. Using this symmetry, we discuss an important property of the self-force induced by a geodesic, and prove that the radiation reaction to the energy E, the angular momentum L and the Carter constant K can be derived by use of a radiative Green function (a half-retarded-minus-half-advanced Green function). However, this is not the end of the story. We find that the orbit does not evolve in a strictly adiabatic manner in general as the energy E, the angular momentum L and the Carter constant K vary on a short time scale. Besides, it is not trivial that the rest of constants do not evolve by the self-force.

In order to complete the calculation of the orbit, we perturbatively integrate the orbital equation with the self-force in Sec.III. We figure out the part which secularly evolves by the self-force, and define the 'adiabatic evolution' of the orbit in the way we can approximately calculate the orbital evolution. By showing the validity of the adiabatic approximation, we propose our new formalism as a conventional tool to predict the gravitational waves detected by the future LISA project.

In this paper, we adopt Boyer-Lindquist coordinates $\{t,\, r,\, \theta,\, \phi\}$, and M, a are the mass and the spin coefficient of the black hole respectively.

II. SELF-FORCE AND SYMMETRY

The purpose of this section is to show a simple method to calculate the self-force. The core idea of this new proposal is to use a symmetry of the background spacetime together with the whole familiy of geodesics.

In Subsec.II A, we first discuss how we can define the family of geodesics. Since we are interested in a particle motion as a target of gravitational wave observation, we only consider geodesics rotating around a Kerr black hole which neither fall into the horizon nor go to infinity.

Subsec.II B discusses some symmetry properties of the Kerr spacetime. We apply these symmetry transformations to the familiy of geodesics and to the self-force induced on a geodesic in Subsec.II C. Using the result of these transformation properties, we discuss a general expression of the evolution equations of the energy, angular momentum, and Carter 'constant' in Subsec.II D, and find that a part of the evolution equation can be evaluated by using the radiative Green function, which has a great computational advantage.

In Subsec.II E, we give some comments about practical issues seriously discussed in the self-force problem.

A. Geodesics around a Kerr black hole

A general geodesic satisfies the equations

$$\left(\frac{dr}{d\lambda}\right)^2 = [(r^2+a^2)E - aL]^2 - \Delta(r^2+K)\,, \tag{2.1}$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = -(aE\sin\theta - L\mathrm{cosec}\theta)^2 - a^2\cos^2\theta + K\,, \tag{2.2}$$

$$\frac{dt}{d\lambda} = \frac{1}{\Delta}\left(\Sigma^2 E - 2aMrL\right)\,, \tag{2.3}$$

$$\frac{d\phi}{d\lambda} = \frac{1}{\Delta}\left[2aMrE + (\rho^2 - 2Mr)L\mathrm{cosec}^2\theta\right]\,, \tag{2.4}$$

‡ We adopt the definition of the Carter constant in Ref. [6].

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \quad \Delta \;=\; r^2 - 2Mr + a^2\,, \tag{2.5}$$
$$\Sigma^2 = (r^2+a^2)\rho^2 + 2a^2Mr\sin^2\theta\,. \tag{2.6}$$

Here we use λ as an orbital parameter related with the proper time τ by $\tau = \int_0^\lambda d\lambda\rho^2$.

We consider the case that the radial motion is bounded in $r_1 < r < r_2$. One can integrate (2.1), and we have a radial periodic solution with respect to λ with the period

$$\lambda_r = 2\int_{r_1}^{r_2} dr \frac{1}{\sqrt{[(r^2+a^2)E - aL]^2 - \Delta(r^2+K)}}\,. \tag{2.7}$$

We write the periodic solution of the radial equation as

$$r(\lambda) \;=\; R(E,L,K;\lambda-\bar\lambda_r) \;=\; R(E,L,K;\lambda-\bar\lambda_r+\lambda_r)\,, \tag{2.8}$$

where we set $R(E,L,K;0) = r_1$ (or, equivalently, $R(E,L,K;\lambda_r/2) = r_2$). $\bar\lambda_r$ is the integral constant of the first differential equation (2.1), and we have a periodicity in $\bar\lambda_r$, namely, $\bar\lambda_r \to \bar\lambda_r + n\lambda_r$ does not change the orbit, where n is an arbitrary integer. Because (2.1) is invariant under $\lambda \to -\lambda$, $R(E,L,K;\lambda)$ becomes a symmetric function with respect to λ as

$$R(E,L,K;\lambda) = R(E,L,K;-\lambda)\,. \tag{2.9}$$

When the orbit is radially bounded, the θ motion oscillates symmetrically around $\theta = \pi/2$ in a domain $0 < \theta_1 < \theta < \pi - \theta_1 < \pi$ [6]. Similarily to the radial motion, we have a θ periodic solution with respect to λ by the period

$$\lambda_\theta = 4\int_{\theta_1}^{\pi/2} d\theta \frac{1}{\sqrt{-(aE\sin\theta - L\mathrm{cosec}\theta)^2 - a^2\cos^2\theta + K}}\,. \tag{2.10}$$

We define the solution of (2.2) as

$$\theta(\lambda) \;=\; \Theta(E,L,K;\lambda-\bar\lambda_\theta) \;=\; \Theta(E,L,K;\lambda-\bar\lambda_\theta+\lambda_\theta)\,, \tag{2.11}$$

where we set $\Theta(E,L,K;0) = \theta_1$ (or, equivalently, $\Theta(E,L,K;\lambda_\theta/2) = \pi - \theta_1$). $\bar\lambda_\theta$ is the integral constant in solving (2.2), and we have a periodicity in $\bar\lambda_\theta$ as in the radial equation (2.8). The symmetry property of (2.11) becomes

$$\Theta(E,L,K;\lambda) = \Theta(E,L,K;-\lambda)\,. \tag{2.12}$$

We write the solutions of (2.3) and (2.4) as

$$t(\lambda) = T(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) + \bar t\,, \tag{2.13}$$
$$T(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) = \int_0^\lambda \frac{d\lambda}{\Delta}\left(\Sigma^2E - 2aMrL\right)\,, \tag{2.14}$$
$$\phi(\lambda) = \Phi(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) + \bar\phi\,, \tag{2.15}$$
$$\Phi(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) = \int_0^\lambda \frac{d\lambda}{\Delta}\left[2aMrE + (\rho^2 - 2Mr)L\mathrm{cosec}^2\theta\right]\,, \tag{2.16}$$

where $\bar t$ and $\bar\phi$ are the integral constants. We have addition formulae of $T(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)$ and $\Phi(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)$ as

$$T(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) = T(E,L,K,\bar\lambda_r-\lambda_x,\bar\lambda_\theta-\lambda_x;\lambda-\lambda_x) + T(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda_x)\,, \tag{2.17}$$
$$\Phi(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) = \Phi(E,L,K,\bar\lambda_r-\lambda_x,\bar\lambda_\theta-\lambda_x;\lambda-\lambda_x) + \Phi(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda_x)\,. \tag{2.18}$$

By using (2.9) and (2.12), the symmetry properties of (2.13) and (2.15) become

$$T(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) \;=\; -T(E,L,K,-\bar\lambda_r,-\bar\lambda_\theta;-\lambda)\,, \quad \Phi(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda) \;=\; -\Phi(E,L,K,-\bar\lambda_r,-\bar\lambda_\theta;-\lambda)\,. \tag{2.19}$$

B. t- and ϕ-translation and Geodesic Preserving Symmetry

As is well-known, a Kerr spacetime has t- and ϕ-translation symmetry. Applying the coordinate transformation as

$$t = t' + t_s \,,\; r = r' \,,\; \theta = \theta' \,,\; \phi = \phi' + \phi_s \,, \tag{2.20}$$

we have a Kerr metric of the same mass and spin parameter with the new coordinates $\{t',\, r',\, \theta',\, \phi'\}$.

We consider to apply this transformation to a geodesic with a λ-translation, $\lambda' = \lambda - \lambda_x$, where λ_x is an arbitrary constant. Using (2.17) and (2.18), we have another geodesic in a new coordinate system as

$$r'(\lambda') = R(E, L, K; \lambda' - \bar{\lambda}'_r) \,, \quad \theta'(\lambda') = \Theta(E, L, K; \lambda' - \bar{\lambda}'_\theta) \,, \tag{2.21}$$

$$t'(\lambda') = T(E, L, K, \bar{\lambda}'_r, \bar{\lambda}'_\theta; \lambda') + \bar{t}' \,, \quad \phi'(\lambda') = \Phi(E, L, K, \bar{\lambda}'_r, \bar{\lambda}'_\theta; \lambda') + \bar{\phi}' \,, \tag{2.22}$$

where the orbital constants, $\{\bar{\lambda}'_r, \bar{\lambda}'_\theta, \bar{t}', \bar{\phi}'\}$, become

$$\bar{\lambda}'_r = \bar{\lambda}_r - \lambda_x \,, \quad \bar{\lambda}'_\theta = \bar{\lambda}_\theta - \lambda_x \,, \tag{2.23}$$

$$\bar{t}' = \bar{t} + T(E, L, K, \bar{\lambda}_r, \bar{\lambda}_\theta; \lambda_x) - t_s \,, \quad \bar{\phi}' = \bar{\phi} + \Phi(E, L, K, \bar{\lambda}_r, \bar{\lambda}_\theta; \lambda_x) - \phi_s \,. \tag{2.24}$$

One sees that the orbital constants, E, L and K, are invarinat under this transformation. We can set $\bar{t}'$, $\bar{\phi}'$ and either $\bar{\lambda}'_r$ or $\bar{\lambda}'_\theta$ arbitrarily by an appropriate choice of t_s, ϕ_s and λ_x.

One cannot set both $\bar{\lambda}'_r$ and $\bar{\lambda}'_\theta$ zero at the same time since we have only one constant λ_x to fix. However, because of the periodicity of the radial function (2.8), one can replace $\bar{\lambda}_r$ and $\bar{\lambda}_\theta$ by numbers congruent to $\bar{\lambda}_r$ and $\bar{\lambda}_\theta$ modulo λ_r and λ_θ respectively, i.e. $\bar{\lambda}_r + n_r\lambda_r$ and $\bar{\lambda}_\theta + n_\theta\lambda_\theta$ where n_r and n_θ are arbitrary integers. Using this freedom, we set $\lambda_x = \bar{\lambda}_r - n_r\lambda_r$, then we obtain $\bar{\lambda}'_r = 0$ and $\bar{\lambda}'_\theta = \bar{\lambda}_\theta - \bar{\lambda}_r + n_r\lambda_r + n_\theta\lambda_\theta$. When the ratio of λ_r and λ_θ is irrational, there is a choice of n_r and n_θ with which $|\bar{\lambda}'_\theta|$ become infinitestimally small, and a geodesic is characterized only by E, L and K. In the following, we assume that the ratio of λ_r and λ_θ is irrational though we do not set $\bar{\lambda}_r$ and $\bar{\lambda}_\theta$ zero for the latter conveience unless stated.

Using this transformation property, one can prove a useful formula of a scalar function geometrically defined along a geodesic. As the geodesic is characterzed by 7 constants, $\{E, L, K, \bar{\lambda}_r, \bar{\lambda}_\theta, \bar{t}, \bar{\phi}\}$, we write the scalar function as $f(E, L, K, \bar{\lambda}_r, \bar{\lambda}_\theta, \bar{t}, \bar{\phi}_0; \lambda)$. We assume that the function is invariant under t- and ϕ-translation, then the function is independ on $\bar{t}$ and $\bar{\phi}$. Since the function is periodic with respect to $\bar{\lambda}_r$ and $\bar{\lambda}_\theta$, one can expand the function with discrete Fourier series, $e^{-i2m\pi\bar{\lambda}_r/\lambda_r - i2n\pi\bar{\lambda}_\theta/\lambda_\theta}$. By applying the λ-translation with $\lambda_x = \lambda$, we finally have

$$f(E, L, K, \bar{\lambda}_r, \bar{\lambda}_\theta, \bar{t}, \bar{\phi}; \lambda) = \sum_{m,n} f^{(m,n)}(E, L, K) \exp\left[i2\pi \left(m\frac{\lambda - \bar{\lambda}_r}{\lambda_r} + n\frac{\lambda - \bar{\lambda}_\theta}{\lambda_\theta} \right) \right] . \tag{2.25}$$

We next consider the symmetry as

$$t = -t' \,,\; r = r' \,,\; \theta = \theta' \,,\; \phi = -\phi' \,. \tag{2.26}$$

By this coordinate transformation, we recover the same line element with the coordinates $\{t',\, r',\, \theta',\, \phi'\}$.

We consider the transformation of a geodesic by this symmetry. Since we change the time direction, we transform the orbital parameter as $\lambda' = -\lambda$. Using (2.9), (2.12) and (2.19), a geodesic is transformed to a new geodesic as

$$r'(\lambda') = R(E, L, K; \lambda' - \bar{\lambda}'_r) \,, \quad \theta'(\lambda') = \Theta(E, L, K; \lambda' - \bar{\lambda}'_\theta) \,, \tag{2.27}$$

$$t'(\lambda') = T(E, L, K, \bar{\lambda}'_r, \bar{\lambda}'_\theta; \lambda') + \bar{t}' \,, \quad \phi'(\lambda') = \Phi(E, L, K, \bar{\lambda}'_r, \bar{\lambda}'_\theta; \lambda') + \bar{\phi}' \,, \tag{2.28}$$

where the orbital constants, $\{\bar{\lambda}'_r, \bar{\lambda}'_\theta, \bar{t}', \bar{\phi}'\}$, become

$$\bar{\lambda}'_r = -\bar{\lambda}_r \,, \quad \bar{\lambda}'_\theta = -\bar{\lambda}_\theta \,, \tag{2.29}$$

$$\bar{t}' = -\bar{t} \,, \quad \bar{\phi}' = -\bar{\phi} \,. \tag{2.30}$$

Since E, L and K are invariant, by using the t- and ϕ- translation symmetry (2.20) in an appropriate manner, the new geodesic becomes equal to the original one. For this reason, we call this Geodesic Preserving Symmetry (hereafter GPS) transformation.

C. Green Function and Self-force

We denote a various scalar/electromagnetic/gravitational Green function by

$$\mathcal{G}(x,z) \;=\; \begin{cases} G(x,z)\,, & \text{scalar}\;\;, \\ G_{\alpha\,\mu}(x,z)dx^\alpha dz^\mu\,, & \text{electromagnetism}\;\;, \\ G_{\alpha\beta\,\mu\nu}(x,z)dx^\alpha dx^\beta dz^\mu dz^\nu\,, & \text{linear gravity}\;\;. \end{cases} \tag{2.31}$$

and consider t- and ϕ-translation and GPS transformation property of it.

Using a symmetric scalar/electromagnetic/gravitational Green function $\mathcal{G}^{sym.}(x,z)$, a retarded and advanced scalar/electromagnetic/gravitational Green function, $\mathcal{G}^{ret./adv.}(x,z)$ can be written as

$$\mathcal{G}^{ret.}(x,z) = 2\theta[\Sigma(x),z]\mathcal{G}^{sym.}(x,z)\,, \tag{2.32}$$

$$\mathcal{G}^{adv.}(x,z) = 2\theta[z,\Sigma(x)]\mathcal{G}^{sym.}(x,z)\,, \tag{2.33}$$

where $\Sigma(x)$ is an arbitrary space-like hypersurface containing x, and $\theta[\Sigma(x),z] = 1-\theta[z,\Sigma(x)]$ is equal to 1 when z lies in the past of $\Sigma(x)$ and vanishes otherwise. The symmetric Green function is invariant under t- and ϕ-translation and GPS transformation because, in its Hadamard construction [4,5], it is described only by geometrically defined bi-tensors invariant under t- and ϕ-translation and GPS transformation.

Under t- and ϕ-translation (2.20), the factor $\theta[\Sigma(x),z]$ is also invariant and we have

$$\mathcal{G}^{ret.}(x',z') \;=\; \mathcal{G}^{ret.}(x,z)\,, \quad \mathcal{G}^{adv.}(x',z') \;=\; \mathcal{G}^{adv.}(x,z)\,. \tag{2.34}$$

On the other hand, GPS transformation (2.26) changes the direction of the time and the factor $\theta[\Sigma(x),z]$ transformes as

$$\theta[\Sigma(x'),z'] \;=\; \theta[z,\Sigma(x)]\,, \quad \theta[z',\Sigma(x')] \;=\; \theta[\Sigma(x),z]\,. \tag{2.35}$$

Thence, by GPS transformation (2.26), a retarded and advanced Green function are transformed to be an advanced and retarded Green function respectively as

$$\mathcal{G}^{ret.}(x',z') \;=\; \mathcal{G}^{adv.}(x,z)\,, \quad \mathcal{G}^{adv.}(x',z') \;=\; \mathcal{G}^{ret.}(x,z)\,. \tag{2.36}$$

We next consider the scalar/electromagnetic/gravitational self-force acting on the particle. Because the field induced by a point particle diverges along the orbit, we need a regularization calculation to derive the self-force [4,5]. Based on the Green function method in calculating the field, an elegant method of regularization was proposed [7], in which the self-force can be directly derived from the field calculated by the so-called R-part of a retarded Green function. The R-part of a retarded and advanced Green function $\mathcal{G}^{R-ret./R-adv.}(x,z)$ is schematically defined as

$$\mathcal{G}^{R-ret./R-adv.}(x,z) \;=\; \mathcal{G}^{ret./adv.}(x,z) - \mathcal{G}^{S}(x,z)\,, \tag{2.37}$$

where $\mathcal{G}^{S}(x,z)$ is the so-called S-part [7]. It is important to note that the R-part of the half-retarded-minus-half-advance Green function becomes a radiative Green function $\mathcal{G}^{rad.}(x,z)$ as

$$\frac{1}{2}\left(\mathcal{G}^{R-ret.}(x,z) - \mathcal{G}^{R-adv.}(x,z)\right) \;=\; \frac{1}{2}\left(\mathcal{G}^{ret.}(x,z) - \mathcal{G}^{adv.}(x,z)\right) \;=\; \mathcal{G}^{rad.}(x,z)\,. \tag{2.38}$$

Similar to the symmetric Green function, the S-part is defined by geometric bi-tensors and is invariant both by t- and ϕ-translation and GPS transformation. Thence the R-part of the retarded and advanced Green function are still invariant under t- and ϕ-translation, and, by GPS transformation, the R-part becomes

$$\mathcal{G}^{R-ret.}(x',z') \;=\; \mathcal{G}^{R-adv.}(x,z)\,, \quad \mathcal{G}^{R-adv.}(x',z') \;=\; \mathcal{G}^{R-ret.}(x,z)\,. \tag{2.39}$$

The scalar/electromagnetic/gravitational self-force is schemetically described as

$$F_\alpha^{R-ret./R-adv.}(\tau) = \lim_{x\to z(\tau)} F_\alpha[\phi^{R-ret./R-adv.}](x)\,, \tag{2.40}$$

$$\phi^{R-ret./R-adv.}(x) = \int d\tau G^{R-ret./R-adv.}(x,z(\tau))S(z(\tau))\,, \tag{2.41}$$

where $\phi^{R-ret./R-adv.}$ is the R-part of a scalar/electromagnetic/gravitational potential using the R-part of a retarded/advanced Green function, and we note $^{R-ret./R-adv.}$ to the self-force to emphasize that it is derived using the

R-part of the retarded/advanced Green function. $S(z(\tau))$ is the source term defined along the orbit. We assume that the tensor differential operator $F_\alpha[](x)$ is defined to satisfy the normalization condition as $F_\alpha[](z(\tau))v^\alpha(\tau)=0$.

We assume that the self-force is weak and the orbit can be approximated to be a geodesic at each instant of time. Using this approximation, we consider to calculate the self-force induced by a geodesic. We write the self-force as a vector function of the orbital constants E, L, K, $\bar\lambda_r$, $\bar\lambda_\theta$, $\bar t$, $\bar\phi$ and the orbital parameter λ as

$$F_\alpha^{R-ret./R-adv.} = F_\alpha^{R-ret./R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta,\bar t,\bar\phi;\lambda)\,. \tag{2.42}$$

In general, 4-velocity and a self-force transform as

$$v^{\alpha'} = \frac{dx'^\alpha}{d\tau'}(\tau') = \left(\frac{\partial x'^\alpha}{\partial x^\beta}\right)\left(\frac{d\tau}{d\tau'}\right)v^\beta(\tau)\,, \tag{2.43}$$

$$f_{\alpha'} = \frac{D}{d\tau'}v_{\alpha'}(\tau') = \left(\frac{d\tau}{d\tau'}\right)\frac{D}{d\tau}\left[\left(\frac{\partial x^\beta}{\partial x'^\alpha}\right)\left(\frac{d\tau}{d\tau'}\right)v_\beta\right](\tau)\,. \tag{2.44}$$

Using these transformation rules, we apply t- and ϕ-translation (2.20) and GPS transformation (2.26) to the self-force induced by a geodesic (2.42).

We first consider t- and ϕ-translation (2.20) with $\lambda_x=0$. Applying the coordinate transformation, we have

$$F_{\alpha'}^{R-ret./R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta,\bar t+t_s,\bar\phi+\phi_s;\lambda) = F_\alpha^{R-ret./R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta,\bar t,\bar\phi;\lambda)\,. \tag{2.45}$$

Since the metric is invariant under the transformation, we have $F_{\alpha'}^{R-ret./R-adv.} = F_\alpha^{R-ret./R-adv.}$, thus, the self-force does not depend on $\bar t$ and $\bar\phi$. In the following, we write the self-force as $F_\alpha^{R-ret./R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)$.

Finally we consider GPS transformation of the self-force. Noting (2.39), (2.26) transforms the self-force as

$$F_\alpha^{R-adv.}(E,L,K,-\bar\lambda_r,-\bar\lambda_\theta;-\lambda) = (-1)^s F_\alpha^{R-ret.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)\,, \tag{2.46}$$

$$F_\alpha^{R-ret.}(E,L,K,-\bar\lambda_r,-\bar\lambda_\theta;-\lambda) = (-1)^s F_\alpha^{R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)\,, \tag{2.47}$$

where $s=1$ for $\alpha=t,\phi$, and $s=0$ for $\alpha=r,\theta$.

D. Evolution of the energy, angular momentum and Carter 'constant'

As we use λ as an orbital parameter, we consider the λ derivative of these 'constants' as

$$\left[\frac{d}{d\lambda}E\right]^{R-ret./R-adv.} = -\rho^2 F_t^{R-ret./R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)\,, \tag{2.48}$$

$$\left[\frac{d}{d\lambda}L\right]^{R-ret./R-adv.} = \rho^2 F_\phi^{R-ret./R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)\,, \tag{2.49}$$

$$\left[\frac{d}{d\lambda}K\right]^{R-ret./R-adv.} = \left[2(r^2+a^2)\rho^2 v^t F_t^{R-ret./R-adv.} + 2\rho^4 v^r F_r^{R-ret./R-adv.}\right](E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)\,. \tag{2.50}$$

Since these are scalar functions defined along a geodesic, we can apply the formula (2.25) to (2.48), (2.49) and (2.50), and we obtain

$$\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-ret./R-adv.} = \sum_{m,n}\dot{\mathcal{E}}^{R-ret./R-adv.(m,n)}(E,L,K)\exp\left[i2\pi\left(m\frac{\lambda-\bar\lambda_r}{\lambda_r}+n\frac{\lambda-\bar\lambda_\theta}{\lambda_\theta}\right)\right]\,, \tag{2.51}$$

where we denote E, L and K by $\mathcal{E}$. By the reality condition, we have $(\dot{\mathcal{E}}^{R-ret./R-adv.\,(m,n)})^* = \dot{\mathcal{E}}^{R-ret./R-adv.\,(-m,-n)}$, where * means we take the complex conjugation operation.

4-velocity of a geodesic transforms by (2.26) as

$$v^{\alpha'}(E,L,K,-\bar\lambda_r,-\bar\lambda_\theta;-\lambda) = (-1)^s v^\alpha(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)\,, \tag{2.52}$$

where $s=0$ for $\alpha=t,\phi$, and $s=1$ for $\alpha=r,\theta$. Using (2.47) and (2.52), (2.48), (2.49) and (2.50) transform as

$$\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-adv./R-ret.}(E,L,K,-\bar\lambda_r,-\bar\lambda_\theta;-\lambda) = -\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-ret./R-adv.}(E,L,K,\bar\lambda_r,\bar\lambda_\theta;\lambda)\,. \tag{2.53}$$

We consider the evolution euations, (2.48), (2.49) and (2.50), averaged at two orbital points characterized as $z(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta,\bar{t},\bar{\phi};\lambda)$ and $z(E,L,K,-\bar{\lambda}_r,-\bar{\lambda}_\theta,\bar{t}',\bar{\phi}';-\lambda)$. Using (2.53), one finds the evolution equations are described by the radiative Green function (2.38) instead of the R-part of the retarded/advanced Green function as

$$\begin{aligned}
&\frac{1}{2}\left\{\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-ret.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)+\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-ret.}(E,L,K,-\bar{\lambda}_r,-\bar{\lambda}_\theta;-\lambda)\right\} \\
&\qquad = -\frac{1}{2}\left\{\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-adv.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)+\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-adv.}(E,L,K,-\bar{\lambda}_r,-\bar{\lambda}_\theta;-\lambda)\right\} \\
&\qquad = \left[\frac{d}{d\lambda}\mathcal{E}\right]^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)\,, \qquad (2.54)
\end{aligned}$$

$$\left[\frac{d}{d\lambda}E\right]^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda) \;=\; -\rho^2 F_t^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)\,, \qquad (2.55)$$

$$\left[\frac{d}{d\lambda}L\right]^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda) \;=\; \rho^2 F_\phi^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)\,, \qquad (2.56)$$

$$\left[\frac{d}{d\lambda}K\right]^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda) \;=\; \left[2(r^2+a^2)\rho^2 v^t F_t^{rad.}+2\rho^4 v^r F_r^{rad.}\right](E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)\,, \qquad (2.57)$$

where $F_\alpha^{rad.}$ is the self-force calculated by the radiative Green function (2.38).

We write

$$\left[\frac{d}{d\lambda}\mathcal{E}\right]^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)=\sum_{m,n}\dot{\mathcal{E}}^{rad.\,(m,n)}(E,L,K)\exp\left[i2\pi\left(m\frac{\lambda-\bar{\lambda}_r}{\lambda_r}+n\frac{\lambda-\bar{\lambda}_\theta}{\lambda_\theta}\right)\right]. \qquad (2.58)$$

then we have

$$\begin{aligned}
\frac{1}{2}\left(\dot{\mathcal{E}}^{R-ret.\,(m,n)}+\dot{\mathcal{E}}^{R-ret.\,(-m,-n)}\right)(E,L,K) &= -\frac{1}{2}\left(\dot{\mathcal{E}}^{R-adv.\,(m,n)}+\dot{\mathcal{E}}^{R-adv.\,(-m,-n)}\right)(E,L,K) \\
&= \dot{\mathcal{E}}^{rad.\,(m,n)}(E,K,L)\,. \qquad (2.59)
\end{aligned}$$

Thus, half of the expansion coefficients of the evolution equations can be derived by using the radiative Green function.

We comment that, when the ratio of λ_r and λ_θ is irrational, our formula genelarizes the result in Ref. [8], in which it is proven that radiation reaction to the energy and the angular momentum along a whole geodesic can be derived by a self-force calculated by a radiative Green function[§]. By (2.51), the radiation reaction averaged per unit λ to the energy, angular momentum and Carter 'constant' becomes

$$\lim_{\lambda\to\infty}\frac{1}{2\lambda}\int_{-\lambda}^{\lambda}d\lambda\frac{d}{d\lambda}\mathcal{E}^{R-ret.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)=\dot{\mathcal{E}}^{R-ret.\,(0,0)}(E,L,K)\,. \qquad (2.60)$$

By (2.59), (2.60) agrees with the calculation using the radiative Green function as

$$\lim_{\lambda\to\infty}\frac{1}{2\lambda}\int_{-\lambda}^{\lambda}d\lambda\frac{d}{d\lambda}\mathcal{E}^{rad.}(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)=\dot{\mathcal{E}}^{R-ret.\,(0,0)}(E,L,K)\,. \qquad (2.61)$$

It is notable that the dependence of both $\bar{\lambda}_r$ and $\bar{\lambda}_\theta$ vanishes in the end.

[§]We also note that our formalism specifies the case that the orbit inducing the self-force can approximated by a geodesic, while the formula in Ref. [8] applies to a general orbit in scalar/electromagnetic case.

E. Practical issues in calculating the radiative potential

The primary problem in a Kerr case is we have no conventional method to calculate a electromagnetic potential and a metric perturbation induced by a point particle [**]. The construction of an inhomogeneous solution is unknown in general, however, a very simple method to derive homogeneous solutions was proposed [10]. In Ref. [10], it was also discussed the derivation of the retarded and advanced Green functions as infinite sums of homogeneous solutions, which gives a correct metric perturbation only outside the source. For example, when the particle moves in the radial domain $r_{min} < r < r_{max}$, the metric perturbation given in Ref. [10] is correct at $r > r_{max}$ and $r < r_{min}$.

Though the prescription in Ref. [10] is insufficient for inhomogeneous Green functions, it gives the correct radiative Green function since it is just a sum of homogeneous solutions. Suppose we calculate the radiative Green function following Ref. [10], it is correct outside the source. However, since it is made as an infinite sum of homogeneous solutions by construction, it satisfies the source-free Einstein equations at every radial domain. Thus, it is a correct radiative Green function in the whole spacetime.

III. PERTURBATIVE EVOLUTION OF AN ORBIT

To make a definite discussion, we consider that, at $\lambda < 0$, the particle moves along a geodesic characterized by the constants, $\mathcal{E} = \mathcal{E}_0$, $\bar\lambda_r = \bar\lambda_{r0}$, $\bar\lambda_\theta = \bar\lambda_{\theta 0}$, $\bar t = \bar t_0$ and $\bar\phi = \bar\phi_0$, and that the self-force begins to act on the orbit when $\lambda > 0$, and deviate from the initial geodesic. In this section, we discuss the deviation of the initial geodesic in a perturbative manner. We define μ as the charge or the mass of the orbiting particle normalized by the mass of the background black hole, and we consider μ is an infinitestimally small value as an index of the perturbation.

In order to see how the orbit evolves by the self-force, we first consider (2.51). We define the deviation of $\mathcal{E}$ from the initial value $\mathcal{E}_0$ by $\delta\mathcal{E}$. Because we only consider the self-force induced by a geodesic in deriving (2.51), we can consistenly derive the evolution of $\delta\mathcal{E}$ only when $\delta\mathcal{E} = O(\mu^\alpha)$, $\alpha > 0$. The evolution of $\delta\mathcal{E}$ becomes

$$\delta\mathcal{E}(E_0, L_0, K_0, \bar\lambda_{r0}, \bar\lambda_{\theta 0}; \lambda) = \lambda \dot{\mathcal{E}}^{R-ret.}(E_0, L_0, K_0) + \sum_{m,n} \mathcal{E}^{R-ret.\,(m,n)}(E_0, L_0, K_0) e^{i2\pi\left(m\frac{\lambda - \bar\lambda_{r0}}{\lambda_r} + n\frac{\lambda - \bar\lambda_{\theta 0}}{\lambda_\theta}\right)}\,, \tag{3.1}$$

where the coefficient of the linearly growing term is determined as

$$\dot{\mathcal{E}}^{R-ret.} = \dot{\mathcal{E}}^{R-ret.\,(0,0)}\,. \tag{3.2}$$

One sees that (3.1) consists of two parts; the secular part and the oscillating part. The oscillating part stays $O(\mu^1)$ at any λ. On the other hand, the secular part grows linearly by λ, thus, one can consistently derive $\delta\mathcal{E}$ only when λ is $O(\mu^\alpha)$, $0 \geq \alpha > -1$.

Because of this oscillating term, one cannot say that the orbit evolves adiabatically in an exact sense. The oscillating part shows the interaction of the orbit and the 'heat bath' of radiation. In the time scale of the order $O(\mu^0)$, the orbit just exchanges the energy, angular momentum with the 'heat bath' and they increase and decrease in the equal rate. In the long time scale of the order $O(\mu^\alpha)$, $0 > \alpha > -1$, the energy and angular momentum reserved in the 'heat bath' escape into the horizon or away to infinity and the orbital energy and angular momentum tend to flow out to the 'heat bath'. Thus, as described by the secular part of (3.1), the orbital energy and angular momentum decrease linearly by λ.

Though we do not have an adiabatic evolution in an exact sense, we show that the secular part of the 'constants' $\mathcal{E}$ becomes dominant over the oscillating part. If the same thing happens to the rest of 'constants', it seems possible to define an 'adiabatic' evolution of the orbit in an approximate sense. For this purpose, we discuss an orbital evolution in a perturbative manner. We first consider the evolution of r and θ coordinates in Subsec.III A, then, t and ϕ coordinates in Subsec.III B. Subsec.III C gives a plausible definition of an 'adiabatic' evolution of the orbit, which approximates the exact orbital evolution by a self-force. Subsec.III D concludes the section with a discussion of a gauge dependence of an 'adiabatic' evolution which appears only in gravitational case.

We define an orbit evolving by a self-force as

$$t(\lambda) = t_0(\lambda) + \delta t(\lambda)\,, \quad r(\lambda) = r_0(\lambda) + \delta r(\lambda)\,, \tag{3.3}$$

$$\theta(\lambda) = \theta_0(\lambda) + \delta\theta(\lambda)\,, \quad \phi(\lambda) = \phi_0(\lambda) + \delta\phi(\lambda)\,, \tag{3.4}$$

where $\{t_0, r_0, \theta_0, \phi_0\}$ is the initial geodesic. For the latter convenience, we define a family of geodesics as

$$t(\lambda) = T(E, L, K, \bar\lambda_r, \bar\lambda_\theta; \lambda) + \bar t\,, \quad r(\lambda) = R(E, L, K; \lambda - \bar\lambda_r)\,, \tag{3.5}$$

$$\theta(\lambda) = \Theta(E, L, K; \lambda - \bar\lambda_r)\,, \quad \phi(\lambda) = \Phi(E, L, K, \bar\lambda_r, \bar\lambda_\theta; \lambda) + \bar\phi\,, \tag{3.6}$$

[**] Recently, some ideas to calculate the vector potential and the metric perturbation induced by a point source were proposed [9].

A. r-motion and θ-motion

Instead of integrating the equation of motion $Dv^\alpha/d\tau = F^\alpha$, we consider to integrate (2.1) and (2.2) to derive the motion of r and θ coodinates. For a convenience, we write (2.1) and (2.2) as

$$\left(\frac{dr}{d\lambda}\right)^2 = V(\mathcal{E}, r)\,, \quad \left(\frac{d\theta}{d\lambda}\right)^2 = U(\mathcal{E}, \theta)\,, \tag{3.7}$$

where $\mathcal{E} = \mathcal{E}_0 + \delta\mathcal{E}$, $r = r_0 + \delta r$ and $\theta = \theta_0 + \delta\theta$.

Taking the leading order deviation from the initial geodesic, (3.7) become

$$2\frac{d\delta r}{d\lambda}\frac{dr_0}{d\lambda} = V(\mathcal{E}_0, r_0)_{,i}\delta\mathcal{E}^i + V(\mathcal{E}_0, r_0)_{,r}\delta r\,, \quad 2\frac{d\delta\theta}{d\lambda}\frac{d\theta_0}{d\lambda} = U(\mathcal{E}_0, \theta_0)_{,i}\delta\mathcal{E}^i + U(\mathcal{E}_0, r_0)_{,\theta}\delta\theta\,, \tag{3.8}$$

where we denote $f_{,i}\delta\mathcal{E}^i = f_{,E}\delta E + f_{,L}\delta L + f_{,K}\delta K$. Using $2d^2r_0/d\lambda^2 = V(\mathcal{E}_0, r_0)_{,r}$ and $2d^2\theta_0/d\lambda^2 = U(\mathcal{E}_0, \theta_0)_{,\theta}$, we have

$$\frac{d}{d\lambda}\left(\frac{\delta r}{\frac{dr_0}{d\lambda}}\right) = \frac{1}{2}\frac{V_{,i}}{V}(\mathcal{E}_0, r_0)\delta\mathcal{E}^i\,, \quad \frac{d}{d\lambda}\left(\frac{\delta\theta}{\frac{d\theta_0}{d\lambda}}\right) = \frac{1}{2}\frac{U_{,i}}{U}(\mathcal{E}_0, \theta_0)\delta\mathcal{E}^i\,. \tag{3.9}$$

The differential equations (3.9) have singularies because $dr_0/d\lambda$ and $V(\mathcal{E}_0, r_0)$ vanish at $\lambda = \bar{\lambda}_r + (n/2)\lambda_r$, and $d\theta_0/d\lambda$ and $U(\mathcal{E}_0, \theta_0)$ vanish at $\lambda = \bar{\lambda}_\theta + (n/2)\lambda_\theta$, where n is an integer.

One must integrate (3.9) such that δr and $\delta\theta$ are smooth at the singularities. We formally integrate the differential equations as

$$\delta r = \frac{dr_0}{d\lambda}\bar{V}_i\delta\mathcal{E}^i - \frac{dr_0}{d\lambda}\int_0^\lambda d\lambda \bar{V}_i\frac{d}{d\lambda}\delta\mathcal{E}^i + c_u^{(n)}\frac{dr_0}{d\lambda}\,, \tag{3.10}$$

$$\delta\theta = \frac{d\theta_0}{d\lambda}\bar{U}_i\delta\mathcal{E}^i - \frac{d\theta_0}{d\lambda}\int_0^\lambda d\lambda \bar{U}_i\frac{d}{d\lambda}\delta\mathcal{E}^i + c_v^{(n)}\frac{d\theta_0}{d\lambda}\,, \tag{3.11}$$

where we define $d\bar{V}_i/d\lambda = V_{,i}/2V$ and $d\bar{U}_i/d\lambda = U_{,i}/2U$. Here one must add the integration constants $c_u^{(n)}$ at $\bar{\lambda}_r + (n+1)\lambda_r/2 > \lambda > \bar{\lambda}_r + n\lambda_r/2$, and $c_v^{(n)}$ at $\bar{\lambda}_\theta + (n+1)\lambda_\theta/2 > \lambda > \bar{\lambda}_\theta + n\lambda_\theta/2$, independently for each integer n, such that $\delta r = 0$ and $\delta\theta = 0$ at $\lambda = 0$. and δr and $\delta\theta$ become smooth at the singularities of (3.9).

In order to determine $c_v^{(n)}$ and $v_u^{(n)}$ together with $\bar{V}_i$ and $\bar{U}_i$, we consider the singular structure of (3.9). We write

$$V(\mathcal{E}_0, r) = v(\mathcal{E}_0, r)\Big(r - r_1(\mathcal{E}_0)\Big)\Big(r_2(\mathcal{E}_0) - r\Big)\,, \tag{3.12}$$

$$U(\mathcal{E}_0, \theta) = u(\mathcal{E}_0, \theta)\Big(\theta - \theta_1(\mathcal{E}_0)\Big)\Big(\pi - \theta_1(\mathcal{E}_0) - \theta\Big)\,, \tag{3.13}$$

where $v(\mathcal{E}_0, r)$ is positive at $r_1 < r < r_2$, and $u(\mathcal{E}_0, \theta)$ is positive at $\theta_1 < \theta < \pi - \theta_1$. r_0 and θ_0 of the initial geodesic around the singularities behave as

$$r_0 = \begin{cases} r_1 + \frac{1}{4}v_1(r_2 - r_1)(\lambda - \bar{\lambda}_r - n\lambda_r)^2 + O\Big((\lambda - \bar{\lambda}_r - n\lambda_r)^4\Big)\,, \\ r_2 - \frac{1}{4}v_2(r_2 - r_1)(\lambda - \bar{\lambda}_r - (n + /2)\lambda_r)^2 + O\Big((\lambda - \bar{\lambda}_r - (n + 1/2)\lambda_r)^4\Big)\,. \end{cases} \tag{3.14}$$

$$\theta_0 = \begin{cases} \theta_1 + \frac{1}{4}u_0(\pi - 2\theta_1)(\lambda - \bar{\lambda}_\theta - n\lambda_\theta)^2 + O\Big((\lambda - \bar{\lambda}_\theta - n\lambda_\theta)^4\Big)\,, \\ \pi - \theta_1 - \frac{1}{4}u_0(\pi - 2\theta_1)(\lambda - \bar{\lambda}_\theta - (n + /2)\lambda_\theta)^2 + O\Big((\lambda - \bar{\lambda}_\theta - (n + 1/2)\lambda_\theta)^4\Big)\,. \end{cases} \tag{3.15}$$

where $v_1 = v(\mathcal{E}_0, r_1)$, $v_2 = v(\mathcal{E}_0, r_2)$, $u_0 = u(\mathcal{E}_0, \theta_1) = u(\mathcal{E}_0, \pi - \theta_1)$, and n is an integer. Thus, the singular structure of $\frac{1}{2}\frac{V_{,i}}{V}(\mathcal{E}_0, r_0)$ and $\frac{1}{2}\frac{U_{,i}}{U}(\mathcal{E}_0, \theta_0)$ becomes

$$\frac{1}{2}\frac{V_{,i}}{V}(\mathcal{E}_0, r_0) = \begin{cases} -\frac{2r_{1,i}}{v_1(r_2 - r_1)}(\lambda - \bar{\lambda}_r - n\lambda_r)^{-2} + O\Big((\lambda - \bar{\lambda}_r - n\lambda_r)^0\Big)\,, \\ \frac{2r_{2,i}}{v_2(r_2 - r_1)}(\lambda - \bar{\lambda}_r - (n + /2)\lambda_r)^{-2} + O\Big((\lambda - \bar{\lambda}_r - (n + 1/2)\lambda_r)^0\Big)\,, \end{cases} \tag{3.16}$$

$$\frac{1}{2}\frac{U_{,i}}{U}(\mathcal{E}_0, \theta_0) = \begin{cases} -\frac{2\theta_{1,i}}{u_0(\pi - 2\theta_1)}(\lambda - \bar{\lambda}_\theta - n\lambda_\theta)^{-2} + O\Big((\lambda - \bar{\lambda}_\theta - n\lambda_\theta)^0\Big)\,, \\ -\frac{2\theta_{1,i}}{u_0(\pi - 2\theta_1)}(\lambda - \bar{\lambda}_\theta - (n + /2)\lambda_\theta)^{-2} + O\Big((\lambda - \bar{\lambda}_\theta - (n + 1/2)\lambda_\theta)^0\Big)\,. \end{cases} \tag{3.17}$$

We define regularization functions as

$$
\begin{aligned}
V_i^{\dagger}(\mathcal{E}_0, \lambda - \bar{\lambda}_{r0}) = i\frac{2\pi}{\lambda_r(r_2 - r_1)}\Bigg[\frac{r_{1,i}}{v_1}\left(\frac{1}{e^{i2\pi(\lambda-\bar{\lambda}_{r0})/\lambda_r} - 1} - \frac{1}{e^{-i2\pi(\lambda-\bar{\lambda}_{r0})/\lambda_r} - 1}\right) \\
+\frac{r_{2,i}}{v_2}\left(\frac{1}{e^{i2\pi(\lambda-\bar{\lambda}_{r0})/\lambda_r} + 1} - \frac{1}{e^{-i2\pi(\lambda-\bar{\lambda}_{r0})/\lambda_r} + 1}\right)\Bigg],
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
U_i^{\dagger}(\mathcal{E}_0, \lambda - \bar{\lambda}_{\theta 0}) = i\frac{2\pi}{\lambda_\theta(r_2 - r_1)}\frac{\theta_{1,i}}{u_0}\Bigg[\left(\frac{1}{e^{i2\pi(\lambda-\bar{\lambda}_{\theta 0})/\lambda_\theta} - 1} - \frac{1}{e^{-i2\pi(\lambda-\bar{\lambda}_{\theta 0})/\lambda_\theta} - 1}\right) \\
-\left(\frac{1}{e^{i2\pi(\lambda-\bar{\lambda}_{\theta 0})/\lambda_\theta} + 1} - \frac{1}{e^{-i2\pi(\lambda-\bar{\lambda}_{\theta 0})/\lambda_\theta} + 1}\right)\Bigg].
\end{aligned} \tag{3.19}
$$

One can see that $dV_i^{\dagger}/d\lambda$ and $dU^{\dagger}/d\lambda$ have the same singular structures as $V_{,i}/2V$ and $U_{,i}/2U$, thus, $V_{,i}/2V - dV_i^{\dagger}/d\lambda$ and $U_{,i}/2U - dU^{\dagger}/d\lambda$ are regular and periodic with the period λ_r and λ_θ. One can expand these differences with discrete Fourier series $e^{i2\pi(\lambda-\bar{\lambda}_{r0})/\lambda_r}$ and $e^{i2\pi(\lambda-\bar{\lambda}_{\theta 0})/\lambda_\theta}$, and one can integrate in a regular manner. We formally write these integrations as

$$
\left[\bar{V}_i - V_i^{\dagger}\right](\mathcal{E}_0, \lambda - \bar{\lambda}_{r0}) = (\lambda - \bar{\lambda}_{r0})\dot{V}_i + \sum_n V_i^{(n)}(\mathcal{E}_0) e^{i2\pi n\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r}}, \tag{3.20}
$$

$$
\left[\bar{U}_i - U_i^{\dagger}\right](\mathcal{E}_0, \lambda - \bar{\lambda}_{\theta 0}) = (\lambda - \bar{\lambda}_{\theta 0})\dot{U}_i + \sum_n U_i^{(n)}(\mathcal{E}_0) e^{i2\pi n\frac{\lambda-\bar{\lambda}_{\theta 0}}{\lambda_\theta}}, \tag{3.21}
$$

where the coefficients of linearly growing terms are

$$
\dot{V}_i(\mathcal{E}_0) = \frac{1}{\lambda_r}\int_0^{\lambda_r} d\lambda\left(\frac{V_{,i}}{2V} - \frac{dV_i^{\dagger}}{d\lambda}\right), \quad \dot{U}_i(\mathcal{E}_0) = \frac{1}{\lambda_\theta}\int_0^{\lambda_\theta} d\lambda\left(\frac{U_{,i}}{2U} - \frac{dU_i^{\dagger}}{d\lambda}\right). \tag{3.22}
$$

We note that there is an ambiguity in adding integral constants, which the finial result does not depend on.

Using (3.18), (3.19), (3.20) and (3.21), the first terms of (3.10) and (3.11) can be separeted as

$$
\frac{dr_0}{d\lambda}\left([\bar{V}_i - V_i^{\dagger}]\delta\mathcal{E}^i + V_i^{\dagger}\delta\mathcal{E}^i\right), \quad \frac{d\theta_0}{d\lambda}\left([\bar{U}_i - U_i^{\dagger}]\delta\mathcal{E}^i + U_i^{\dagger}\delta\mathcal{E}^i\right).
$$

Now we can see that we successfully regularize the first terms of (3.10) and (3.11) since the divergent behavior of $V_i^{\dagger}$ and $U_i^{\dagger}$ and the regular behavior of $dr_0/d\lambda$ and $d\theta_0/d\lambda$ cancel each other at the singularities of (3.9) as

$$
\frac{dr_0}{d\lambda}V_i^{\dagger} = \begin{cases} r_{1,i} + O\Big((\lambda - \bar{\lambda}_{r0} - n\lambda_r)^2\Big), \\ r_{2,i} + O\Big((\lambda - \bar{\lambda}_{r0} - (n+1/2)\lambda_r)^2\Big), \end{cases} \tag{3.23}
$$

$$
\frac{d\theta_0}{d\lambda}U_i^{\dagger} = \begin{cases} \theta_{1,i} + O\Big((\lambda - \bar{\lambda}_{\theta 0} - n\lambda_\theta)^2\Big), \\ -\theta_{1,i} + O\Big((\lambda - \bar{\lambda}_{\theta 0} - (n+1/2)\lambda_\theta)^2\Big). \end{cases} \tag{3.24}
$$

We can further rewrite the first terms of (3.10) and (3.11) using $R_{,i}(\mathcal{E}_0, \lambda - \bar{\lambda}_{r0})$ and $\Theta_{,i}(\mathcal{E}_0, \lambda - \bar{\lambda}_{\theta 0})$. By taking the $\mathcal{E}$ derivative of the geodesic equation, we have

$$
\frac{d}{d\lambda}\left(\frac{R_{,i}}{\frac{dr_0}{d\lambda}}\right) = \frac{1}{2}\frac{V_{,i}}{V}(\mathcal{E}_0, r_0), \quad \frac{d}{d\lambda}\left(\frac{\Theta_{,i}}{\frac{d\theta_0}{d\lambda}}\right) = \frac{1}{2}\frac{U_{,i}}{U}(\mathcal{E}_0, \theta_0). \tag{3.25}
$$

These equations have singularities as (3.9), and must be integrated such that $R_{,i}$ and $\Theta_{,i}$ are smooth at the singular points. Using (3.18), (3.19), (3.20) and (3.21), we have

$$
R_{,i}(\mathcal{E}_0, \lambda - \bar{\lambda}_{r0}) = \frac{dr_0}{d\lambda}\left([\bar{V}_i - V_i^{\dagger}] + V_i^{\dagger} + c_i^{(v)}\right), \quad \Theta_{,i}(\mathcal{E}_0, \lambda - \bar{\lambda}_{\theta 0}) = \frac{d\theta_0}{d\lambda}\left([\bar{U}_i - U_i^{\dagger}] + U_i^{\dagger} + c_i^{(u)}\right), \tag{3.26}
$$

where $c_i^{(v)}$ and $c_i^{(u)}$ are finite integral constants. Using the ambiguity of integral constants in evaluating $[\bar{V}_i - V_i^\dagger]$ and $[\bar{U}_i - U_i^\dagger]$, we set $c_i^{(v)} = c_i^{(u)} = 0$. and we have the first terms of (3.10) and (3.11) as

$$R_{,i}(\mathcal{E}_0, \lambda - \bar{\lambda}_{r0})\delta\mathcal{E}^i\,, \quad \Theta_{,i}(\mathcal{E}_0, \lambda - \bar{\lambda}_{\theta 0})\delta\mathcal{E}^i\,. \tag{3.27}$$

Since r- and θ-motion of a geodesic is periodic, we can put

$$R(\mathcal{E}, \lambda - \bar{\lambda}_r) \;=\; \sum_n R^{(n)}(\mathcal{E}) e^{i2\pi n \frac{\lambda - \bar{\lambda}_r}{\lambda_r(\mathcal{E})}}\,, \quad \Theta(\mathcal{E}, \lambda - \bar{\lambda}_r) \;=\; \sum_n \Theta^{(n)}(\mathcal{E}) e^{i2\pi n \frac{\lambda - \bar{\lambda}_\theta}{\lambda_\theta(\mathcal{E})}}\,. \tag{3.28}$$

The $\mathcal{E}$-derivative of R and Θ becomes

$$R_{,i} = \sum_n \left(-i2\pi n \frac{\lambda - \bar{\lambda}_r}{\lambda_r}\frac{\lambda_{r,i}}{\lambda_r} R^{(n)} + R^{(n)}_{,i}\right) e^{i2\pi n \frac{\lambda - \bar{\lambda}_r}{\lambda_r}} \;=\; -(\lambda - \bar{\lambda}_r)\frac{dR}{d\lambda}\frac{\lambda_{r,i}}{\lambda_r} + \sum_n R^{(n)}_{,i} e^{i2\pi n \frac{\lambda - \bar{\lambda}_r}{\lambda_r}}\,, \tag{3.29}$$

$$\Theta_{,i} = \sum_n \left(-i2\pi n \frac{\lambda - \bar{\lambda}_\theta}{\lambda_\theta}\frac{\lambda_{\theta,i}}{\lambda_\theta} \Theta^{(n)} + \Theta^{(n)}_{,i}\right) e^{i2\pi n \frac{\lambda - \bar{\lambda}_\theta}{\lambda_\theta}} \;=\; -(\lambda - \bar{\lambda}_\theta)\frac{d\Theta}{d\lambda}\frac{\lambda_{\theta,i}}{\lambda_\theta} + \sum_n \Theta^{(n)}_{,i} e^{i2\pi n \frac{\lambda - \bar{\lambda}_\theta}{\lambda_\theta}}\,. \tag{3.30}$$

One can see that $R_{,i}$ and $\Theta_{,}$ are dominated by oscillating parts whose amplitude grows linearly in λ. Comparing (3.20), (3.21) and (3.26), we find

$$\dot{V}_i(\mathcal{E}_0) \;=\; -\frac{\lambda_{r,i}}{\lambda_r}(\mathcal{E}_0)\,, \quad \dot{U}_i(\mathcal{E}_0) \;=\; -\frac{\lambda_{\theta,i}}{\lambda_\theta}(\mathcal{E}_0)\,. \tag{3.31}$$

We next discuss the second terms of (3.10) and (3.11). Using (3.18), (3.19) (3.20) and (3.21), we separete the terms as

$$-\frac{dr_0}{d\lambda}\left(\int_0^\lambda d\lambda[\bar{V}_i - V_i^\dagger]\frac{d}{d\lambda}\delta\mathcal{E}^i + \int_0^\lambda d\lambda V_i^\dagger \frac{d}{d\lambda}\delta\mathcal{E}^i\right), \tag{3.32}$$

$$-\frac{d\theta_0}{d\lambda}\left(\int_0^\lambda d\lambda[\bar{U}_i - U_i^\dagger]\frac{d}{d\lambda}\delta\mathcal{E}^i + \int_0^\lambda d\lambda_0 U_i^\dagger \frac{d}{d\lambda}\delta\mathcal{E}^i\right). \tag{3.33}$$

The integrands of the first terms in the brackets are regular and periodic, and one can evaluate the integration in an usual manner as

$$\int_0^\lambda d\lambda[\bar{V}_i - V_i^\dagger]\frac{d}{d\lambda}\delta\mathcal{E}^i = \frac{(\lambda - \bar{\lambda}_{r0})^2}{2}\ddot{V}_e + \sum_{m,n}\left((\lambda - \bar{\lambda}_{r0})\dot{V}_e^{(m,n)} + V_e^{(m,n)}\right) e^{i2\pi\left(m\frac{\lambda - \bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda - \bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,, \tag{3.34}$$

$$\int_0^\lambda d\lambda[\bar{U}_i - U_i^\dagger]\frac{d}{d\lambda}\delta\mathcal{E}^i = \frac{(\lambda - \bar{\lambda}_{\theta 0})^2}{2}\ddot{U}_e + \sum_{m,n}\left((\lambda - \bar{\lambda}_{\theta 0})\dot{U}_e^{(m,n)} + U_e^{(m,n)}\right) e^{i2\pi\left(m\frac{\lambda - \bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda - \bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,. \tag{3.35}$$

The second terms in the brackets could lead to logarithmic divergence at the singular points, $r_0 = r_1, r_2$ and $\theta_0 = \theta_1, \pi - \theta_1$. If we have logarithmic divergence, we have no way to have a smooth evolution of the orbit at the singularities. Thus, one must constraint on the self-force as

$$0 = r_{1,i}\frac{d}{d\lambda}\delta\mathcal{E}^i(\lambda = \bar{\lambda}_{r0} + n\lambda_r) \;=\; r_{2,i}\frac{d}{d\lambda}\delta\mathcal{E}^i(\lambda = \bar{\lambda}_{r0} + (n + 1/2)\lambda_r)\,, \tag{3.36}$$

$$0 = \theta_{1,i}\frac{d}{d\lambda}\delta\mathcal{E}^i(\lambda = \bar{\lambda}_{\theta 0} + n\lambda_\theta) \;=\; \theta_{1,i}\frac{d}{d\lambda}\delta\mathcal{E}^i(\lambda = \bar{\lambda}_{\theta 0} + (n + 1/2)\lambda_\theta)\,. \tag{3.37}$$

Using (2.51), we can re-expand as

$$r_{1,i}\frac{d}{d\lambda}\delta\mathcal{E}^i = \sum_{m\neq 0,n}\dot{\mathcal{E}}_{r1}^{(m,n)}(e^{i2m\pi\frac{\lambda - \bar{\lambda}_{r0}}{\lambda_r}} - 1)e^{i2n\pi\frac{\lambda - \bar{\lambda}_{\theta 0}}{\lambda_\theta}}\,, \tag{3.38}$$

$$r_{2,i}\frac{d}{d\lambda}\delta\mathcal{E}^i = \sum_{m\neq 0,n}\dot{\mathcal{E}}_{r2}^{(m,n)}((-1)^m e^{i2m\pi\frac{\lambda - \bar{\lambda}_{r0}}{\lambda_r}} - 1)e^{i2n\pi\frac{\lambda - \bar{\lambda}_{\theta 0}}{\lambda_\theta}}\,, \tag{3.39}$$

$$\theta_{1,i}\frac{d}{d\lambda}\delta\mathcal{E}^i = \sum_{m,n\neq 0}\dot{\mathcal{E}}_{\theta 1}^{(m,n)}e^{i2m\pi\frac{\lambda - \bar{\lambda}_{r0}}{\lambda_r}}(e^{i2n\pi\frac{\lambda - \bar{\lambda}_{\theta 0}}{\lambda_\theta}} - 1)\,, \tag{3.40}$$

$$-\theta_{1,i}\frac{d}{d\lambda}\delta\mathcal{E}^i = \sum_{m\neq 0,n}\dot{\mathcal{E}}_{\theta 2}^{(m,n)}e^{i2m\pi\frac{\lambda - \bar{\lambda}_{r0}}{\lambda_r}}((-1)^m e^{i2n\pi\frac{\lambda - \bar{\lambda}_{\theta 0}}{\lambda_\theta}} - 1)\,, \tag{3.41}$$

where $\dot{\mathcal{E}}_{r1}^{(m,n)} = r_{1,i}\dot{\mathcal{E}}^{i(m,n)}$, $\dot{\mathcal{E}}_{r2}^{(m,n)} = (-1)^m r_{2,i}\dot{\mathcal{E}}^{i(m,n)}$, $\dot{\mathcal{E}}_{\theta1}^{(m,n)} = \theta_{1,i}\dot{\mathcal{E}}^{i(m,n)}$ and $\dot{\mathcal{E}}_{\theta2}^{(m,n)} = -(-1)^n\theta_{1,i}\dot{\mathcal{E}}^{i(m,n)}$. Using these expansions, we can integrate the second term without logarithmic divergence as

$$\int_0^\lambda d\lambda V_i^\dagger \frac{d}{d\lambda}\delta\mathcal{E}^i = (\lambda - \bar{\lambda}_{r0})\dot{V}_e^\dagger + \sum_{m,n} V_e^{\dagger(m,n)} e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}, \tag{3.42}$$

$$\int_0^\lambda d\lambda U_i^\dagger \frac{d}{d\lambda}\delta\mathcal{E}^i = (\lambda - \bar{\lambda}_{\theta0})\dot{U}_e^\dagger + \sum_{m,n} U_e^{\dagger(m,n)} e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}. \tag{3.43}$$

It is notable that we have only linearly growing terms, thus, (3.34) and (3.35) dominate over (3.42) and (3.43) at $\lambda \sim O(\mu^\alpha)$, $0 > \alpha > -1/2$.

Using (3.26), one can rewrite the second terms of (3.10) and (3.11) as

$$-\frac{dr_0}{d\lambda}\int_0^\lambda d\lambda \frac{R_{,i}}{\frac{dr_0}{d\lambda}}\frac{d}{d\lambda}\delta\mathcal{E}^i, \quad -\frac{d\theta_0}{d\lambda}\int_0^\lambda d\lambda \frac{\Theta_{,i}}{\frac{d\theta_0}{d\lambda}}\frac{d}{d\lambda}\delta\mathcal{E}^i. \tag{3.44}$$

By the constraints on the self-force (3.36) and (3.37), the integrands of (3.44) have no singularity, and we can formally write as

$$\frac{R_{,i}}{\frac{dr_0}{d\lambda}}\frac{d}{d\lambda}\delta\mathcal{E}^i = \sum_{m,n}\left((\lambda-\bar{\lambda}_{r0})\dot{R}_e^{(m,n)} + R_e^{(m,n)}\right)e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}, \tag{3.45}$$

$$\frac{\Theta_{,i}}{\frac{d\theta_0}{d\lambda}}\frac{d}{d\lambda}\delta\mathcal{E}^i = \sum_{m,n}\left((\lambda-\bar{\lambda}_{\theta0})\dot{\Theta}_e^{(m,n)} + \Theta_e^{(m,n)}\right)e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}, \tag{3.46}$$

where the coefficients of linearly growing terms become

$$\dot{R}_e^{(m,n)} = -\frac{\lambda_{r,i}}{\lambda_r}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}, \quad \dot{\Theta}_e^{(m,n)} = -\frac{\lambda_{\theta,i}}{\lambda_\theta}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}. \tag{3.47}$$

We show how to integrate (3.10) and (3.11) such that δr and $\delta\theta$ evolve smoothly, and we find $c_v^{(n)} = c_u^{(n)} = 0$ along this integration procedure. The perturbative evolution of δr and $\delta\theta$ by the self-force is now interpreted as the evolution of the orbital 'constants', $\delta\mathcal{E}^i$, $\delta\bar{\lambda}_r$ and $\delta\bar{\lambda}_\theta$ as

$$\delta r = R_{,i}\delta\mathcal{E}^i + R_{,\bar{\lambda}_r}\delta\bar{\lambda}_r, \tag{3.48}$$

$$\delta\bar{\lambda}_r = \frac{(\lambda-\bar{\lambda}_{r0})^2}{2}\ddot{\lambda}_r + \sum_{m,n}\left((\lambda-\bar{\lambda}_{r0})\dot{\lambda}_r^{(m,n)} + \lambda_r^{(m,n)}\right)e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}, \tag{3.49}$$

$$\delta\theta = \Theta_{,i}\delta\mathcal{E}^i + \Theta_{,\bar{\lambda}_\theta}\delta\bar{\lambda}_\theta, \tag{3.50}$$

$$\delta\bar{\lambda}_\theta = \frac{(\lambda-\bar{\lambda}_{\theta0})^2}{2}\ddot{\lambda}_\theta + \sum_{m,n}\left((\lambda-\bar{\lambda}_{\theta0})\dot{\lambda}_\theta^{(m,n)} + \lambda_\theta^{(m,n)}\right)e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}. \tag{3.51}$$

From (3.47), we have

$$\ddot{\lambda}_r = = -\frac{\lambda_{r,i}}{\lambda_r}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}, \quad \ddot{\lambda}_\theta = -\frac{\lambda_{\theta,i}}{\lambda_\theta}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}, \tag{3.52}$$

$$\dot{\lambda}_r^{(m,n)} = \frac{i}{2\pi\left(\frac{m}{\lambda_r}+\frac{n}{\lambda_\theta}\right)}\frac{\lambda_{r,i}}{\lambda_r}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}, \quad \dot{\lambda}_\theta^{(m,n)} = \frac{i}{2\pi\left(\frac{m}{\lambda_r}+\frac{n}{\lambda_\theta}\right)}\frac{\lambda_{\theta,i}}{\lambda_\theta}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}, \tag{3.53}$$

$$\dot{\lambda}_r^{(0,0)} = R_e^{(0,0)} = \dot{V}_e^{(0,0)} + \dot{V}_e^\dagger, \quad \dot{\lambda}_\theta^{(0,0)} = \Theta_e^{(0,0)} = \dot{U}_e^{(0,0)} + \dot{U}_e^\dagger, \tag{3.54}$$

$$\lambda_r^{(m,n)} = \frac{-i}{2\pi\left(\frac{m}{\lambda_r}+\frac{n}{\lambda_\theta}\right)}R_e^{(m,n)} = V_e^{(m,n)} + V_e^{\dagger(m,n)}, \quad \lambda_\theta^{(m,n)} = \frac{-i}{2\pi\left(\frac{m}{\lambda_r}+\frac{n}{\lambda_\theta}\right)}\Theta_e^{(m,n)} = U_e^{(m,n)} + U_e^{\dagger(m,n)}, \tag{3.55}$$

where the second line and the fourth line are evaluated for $(m,n) \neq (0,0)$, and $\lambda_r^{(0,0)}$, $\lambda_\theta^{(0,0)}$ are determined such that $\delta\bar{\lambda}_0 = \delta\bar{\lambda}_\theta = 0$ at $\lambda = 0$.

We can see that the linear perturbation holds for $\lambda \sim O(\mu^\alpha)$, $0 \geq \alpha > -1/2$. The orbital evolution in this regime has two parts; the one contributed by $\delta\mathcal{E}$, the other by $\delta\bar{\lambda}_r$ and $\delta\bar{\lambda}_\theta$. Since both parts grow quadratically by λ, it is necessary to consider the evolution of $\bar{\lambda}_r$ and $\bar{\lambda}_\theta$ for a correct orbital prediction.

B. t-motion and ϕ-motion

We write (2.3) and (2.4) as

$$\frac{dt}{d\lambda} = X(\mathcal{E}, r, \theta)\,, \quad \frac{d\phi}{d\lambda} = Y(\mathcal{E}, r, \theta)\,. \tag{3.56}$$

The leading deviation from the initial geodesic satisfies

$$\begin{aligned}\frac{d\delta t}{d\lambda} &= X_{,i}(\mathcal{E}_0, r_0, \theta_0)\delta\mathcal{E}^i + X_{,r}(\mathcal{E}_0, r_0, \theta_0)\delta r + X_{,\theta}(\mathcal{E}_0, r_0, \theta_0)\delta\theta \\ &= (X_{,i} + X_{,r}R_{,i} + X_{,\theta}\Theta_{,i})\delta\mathcal{E}^i + X_{,r}R_{,\bar{\lambda}_r}\delta\bar{\lambda}_r + X_{,\theta}\Theta_{,\bar{\lambda}_\theta}\delta\bar{\lambda}_\theta\,,\end{aligned} \tag{3.57}$$

$$\begin{aligned}\frac{d\delta\phi}{d\lambda} &= Y_{,i}(\mathcal{E}_0, r_0, \theta_0)\delta\mathcal{E}^i + Y_{,r}(\mathcal{E}_0, r_0, \theta_0)\delta r + Y_{,\theta}(\mathcal{E}_0, r_0, \theta_0)\delta\theta \\ &= (Y_{,i} + Y_{,r}R_{,i} + Y_{,\theta}\Theta_{,i})\delta\mathcal{E}^i + Y_{,r}R_{,\bar{\lambda}_r}\delta\bar{\lambda}_r + Y_{,\theta}\Theta_{,\bar{\lambda}_\theta}\delta\bar{\lambda}_\theta\,,\end{aligned} \tag{3.58}$$

where we use (3.48) and (3.50). Contrary to (3.9), (3.57) and (3.58) are regular, and we can integrate by parts as

$$\delta t = \bar{X}_i\delta\mathcal{E}^i + \bar{X}_r\delta\bar{\lambda}_r + \bar{X}_\theta\delta\bar{\lambda}_\theta - \int_0^\lambda d\lambda\left(\bar{X}_i\frac{d}{d\lambda}\delta\mathcal{E}^i + \bar{X}_r\frac{d}{d\lambda}\delta\bar{\lambda}_r + \bar{X}_\theta\frac{d}{d\lambda}\delta\bar{\lambda}_\theta\right), \tag{3.59}$$

$$\frac{d}{d\lambda}\bar{X}_i = X_{,i} + X_{,r}R_{,i} + X_{,\theta}\Theta_{,i}\,, \quad \frac{d}{d\lambda}\bar{X}_r = X_{,r}R_{,\bar{\lambda}_r}\,, \quad \frac{d}{d\lambda}\bar{X}_\theta = X_{,\theta}\Theta_{,\bar{\lambda}_\theta}\,, \tag{3.60}$$

$$\delta\phi = \bar{Y}_i\delta\mathcal{E}^i + \bar{Y}_r\delta\bar{\lambda}_r + \bar{Y}_\theta\delta\bar{\lambda}_\theta - \int_0^\lambda d\lambda\left(\bar{Y}_i\frac{d}{d\lambda}\delta\mathcal{E}^i + \bar{Y}_r\frac{d}{d\lambda}\delta\bar{\lambda}_r + \bar{Y}_\theta\frac{d}{d\lambda}\delta\bar{\lambda}_\theta\right), \tag{3.61}$$

$$\frac{d}{d\lambda}\bar{Y}_i = Y_{,i} + Y_{,r}R_{,i} + Y_{,\theta}\Theta_{,i}\,, \quad \frac{d}{d\lambda}\bar{Y}_r = Y_{,r}R_{,\bar{\lambda}_r}\,, \quad \frac{d}{d\lambda}\bar{Y}_\theta = Y_{,\theta}\Theta_{,\bar{\lambda}_\theta}\,. \tag{3.62}$$

We note that $T_{,i}(\mathcal{E}_0, \bar{\lambda}_{r0}, \bar{\lambda}_{\theta 0}; \lambda)$ and $\Phi_{,i}(\mathcal{E}_0, \bar{\lambda}_{r0}, \bar{\lambda}_{\theta 0}; \lambda)$ satisfy

$$\frac{d}{d\lambda}T_{,i} = X_{,i} + X_{,r}R_{,i} + X_{,\theta}\Theta_{,i}\,, \quad \frac{d}{d\lambda}\Phi_{,i} = X_{,i} + X_{,r}R_{,i} + X_{,\theta}\Theta_{,i}\,, \tag{3.63}$$

thus, from (3.60) and (3.62), we have

$$\bar{X}_i = T_{,i}(\mathcal{E}_0, \bar{\lambda}_{r0}, \bar{\lambda}_{\theta 0}) + c^{(x)}\,, \quad \bar{Y}_i = \Phi_{,i}(\mathcal{E}_0, \bar{\lambda}_{r0}, \bar{\lambda}_{\theta 0}) + c^{(y)}\,, \tag{3.64}$$

where $c^{(x)}$ and $c^{(y)}$ are integral constants. Because the orbital evolution does not depend on $c^{(x)}$ and $c^{(y)}$ in the end, we set them zero in this subsection. In the same manner, we have

$$\bar{X}_r = T_{,\bar{\lambda}_r}\,, \quad \bar{X}_\theta = T_{,\bar{\lambda}_\theta}\,, \quad \bar{Y}_r = \Phi_{,\bar{\lambda}_r}\,, \quad \bar{Y}_\theta = \Phi_{,\bar{\lambda}_\theta}\,. \tag{3.65}$$

We define t- and ϕ-motion of a geodesic as

$$T(\mathcal{E}, \bar{\lambda}_r, \bar{\lambda}_\theta; \lambda) = \lambda\dot{T}(\mathcal{E}) + \sum_{m,n} T^{(m,n)}(\mathcal{E})e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_r}{\lambda_r(\mathcal{E})} + n\frac{\lambda-\bar{\lambda}_\theta}{\lambda_\theta(\mathcal{E})}\right)}\,, \tag{3.66}$$

$$\Phi(\mathcal{E}, \bar{\lambda}_r, \bar{\lambda}_\theta; \lambda) = \lambda\dot{\Phi}(\mathcal{E}) + \sum_{m,n} \Phi^{(m,n)}(\mathcal{E})e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_r}{\lambda_r(\mathcal{E})} + n\frac{\lambda-\bar{\lambda}_\theta}{\lambda_\theta(\mathcal{E})}\right)}\,, \tag{3.67}$$

where the linearly growing terms appear since we integrate $X(\mathcal{E}, R, \Theta)$ and $Y(\mathcal{E}, R, \Theta)$ which can be expanded by discrete Fourier-series $e^{i2\pi m(\lambda-\bar{\lambda}_r)/\lambda_r + i2\pi n(\lambda-\bar{\lambda}_\theta)/\lambda_\theta}$. Using (3.64) and (3.65), we have

$$\bar{X}_i = \sum_{m,n}\big(\lambda\dot{X}_i^{(m,n)} + X_i^{(m,n)}\big)e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,, \quad \bar{Y}_i = \sum_{m,n}\big(\lambda\dot{Y}_i^{(m,n)} + Y_i^{(m,n)}\big)e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,, \tag{3.68}$$

$$\bar{X}_r = \sum_{m,n} X_r^{(m,n)}e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,, \quad \bar{Y}_r = \sum_{m,n} Y_r^{(m,n)}e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,, \tag{3.69}$$

$$\bar{X}_\theta = \sum_{m,n} X_\theta^{(m,n)}e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,, \quad \bar{Y}_\theta = \sum_{m,n} Y_\theta^{(m,n)}e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r} + n\frac{\lambda-\bar{\lambda}_{\theta 0}}{\lambda_\theta}\right)}\,, \tag{3.70}$$

where the coefficients become

$$\dot{X}_i^{(m,n)} = \delta_{m=n=0}\dot{T}_{,i} - i2\pi\left(m\frac{\lambda_{r,i}}{\lambda_r^2}+n\frac{\lambda_{\theta,i}}{\lambda_\theta^2}\right)T^{(m,n)}\,, \quad X_i^{(m,n)} \;=\; T_{,i}^{(m,n)} + i2\pi\left(m\frac{\bar{\lambda}_r\lambda_{r,i}}{\lambda_r^2}+n\frac{\bar{\lambda}_\theta\lambda_{\theta,i}}{\lambda_\theta^2}\right)T^{(m,n)}\,, \tag{3.71}$$

$$\dot{Y}_i^{(m,n)} = \delta_{m=n=0}\dot{\Phi}_{,i} - i2\pi\left(m\frac{\lambda_{r,i}}{\lambda_r^2}+n\frac{\lambda_{\theta,i}}{\lambda_\theta^2}\right)\Phi^{(m,n)}\,, \quad Y_i^{(m,n)} \;=\; \Phi_{,i}^{(m,n)} + i2\pi\left(m\frac{\bar{\lambda}_r\lambda_{r,i}}{\lambda_r^2}+n\frac{\bar{\lambda}_\theta\lambda_{\theta,i}}{\lambda_\theta^2}\right)\Phi^{(m,n)}\,, \tag{3.72}$$

$$X_r^{(m,n)} = -m\frac{i2\pi}{\lambda_r}T^{(m,n)}\,, \quad X_\theta^{(m,n)} \;=\; -n\frac{i2\pi}{\lambda_\theta}T^{(m,n)}\,, \quad Y_r^{(m,n)} \;=\; -m\frac{i2\pi}{\lambda_r}\Phi^{(m,n)}\,, \quad Y_\theta^{(m,n)} \;=\; -n\frac{i2\pi}{\lambda_\theta}\Phi^{(m,n)}\,. \tag{3.73}$$

We consider to evaluate the integration of (3.59) and (3.61). We first note

$$\frac{d}{d\lambda}\delta\bar{\lambda}_r = \frac{R_{,i}}{\frac{dr_0}{d\lambda}}\frac{d}{d\lambda}\delta\mathcal{E}^i\,, \quad \frac{d}{d\lambda}\delta\bar{\lambda}_\theta \;=\; \frac{\Theta_{,i}}{\frac{d\theta_0}{d\lambda}}\frac{d}{d\lambda}\delta\mathcal{E}^i\,, \tag{3.74}$$

and the expansion of RHSs are given in (3.42), (3.43) and (3.47). Using these results, the integrands of (3.59) and (3.61) become

$$\sum_{m,n}(\lambda\dot{X}_e^{(m,n)} + X_e^{(m,n)})e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r}+n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}\,, \quad \sum_{m,n}(\lambda\dot{Y}_e^{(m,n)} + Y_e^{(m,n)})e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r}+n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}\,, \tag{3.75}$$

where the coefficients of the linearly growing and oscillating terms become

$$\dot{X}_e^{(m,n)} \;=\; \dot{T}_{,i}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}\,, \quad \dot{Y}_e^{(m,n)} \;=\; \dot{\Phi}_{,i}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}\,. \tag{3.76}$$

In summary, we have t- and ϕ-motion as

$$\delta t = T_{,i}\delta\mathcal{E}^i + T_{,\bar{\lambda}_r}\delta\bar{\lambda}_r + T_{,\bar{\lambda}_\theta}\delta\bar{\lambda}_\theta + \delta\bar{t}\,, \tag{3.77}$$

$$\delta\bar{t} = \frac{\lambda^2}{2}\ddot{t}_0 + \sum_{m,n}(\lambda\dot{t}_0^{(m,n)} + t_0^{(m,n)})e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r}+n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}\,, \tag{3.78}$$

$$\delta\phi = \Phi_{,i}\delta\mathcal{E}^i + \Phi_{,\bar{\lambda}_r}\delta\bar{\lambda}_r + \Phi_{,\bar{\lambda}_\theta}\delta\bar{\lambda}_\theta + \delta\bar{\phi}\,, \tag{3.79}$$

$$\delta\bar{\phi} = \frac{\lambda^2}{2}\ddot{\phi}_0 + \sum_{m,n}(\lambda\dot{\phi}_0^{(m,n)} + \phi_0^{(m,n)})e^{i2\pi\left(m\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r}+n\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\right)}\,, \tag{3.80}$$

where the part of coefficients of $\delta\bar{t}$ and $\delta\bar{\phi}$ are

$$\ddot{t}_0 = -\dot{T}_{,i}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}\,, \quad \ddot{\phi}_0 \;=\; -\dot{\Phi}_{,i}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}\,, \tag{3.81}$$

$$\dot{t}_0^{(m,n)} \;=\; \frac{i}{2\pi\left(\frac{m}{\lambda_r}+\frac{n}{\lambda_\theta}\right)}\dot{T}_{,i}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}\,, \quad \dot{\phi}_0^{(m,n)} \;=\; \frac{i}{2\pi\left(\frac{m}{\lambda_r}+\frac{n}{\lambda_\theta}\right)}\dot{\Phi}_{,i}\dot{\mathcal{E}}^{R-ret.\,(m,n)\,i}\,, \quad (m,n)\neq(0,0)\,. \tag{3.82}$$

As in the case of r- and θ-motion, the linear perturbation holds for $\lambda \sim O(\mu^\alpha)$, $0 \geq \alpha > -1/2$[††], and $\delta\mathcal{E}$, $\delta\bar{\lambda}_r$, $\delta\bar{\lambda}_\theta$, $\delta\bar{t}$ and $\delta\bar{\phi}$ contribute to the orbital evolution equally.

C. Adiabatic Evolution

In Subsec.III A and Subsec.III B, we find the evolution of the orbit by the self-force in a perturbative manner. For the definite discussion, we assume that the self-force begins to act on the orbit at $\lambda = 0$, and find that the perturbative evolution is correct at $\lambda = O(\mu^\alpha)$, $0 \geq \alpha > -1/2$. Using this result, we first consider to rewrite the orbital equation in a numerically convenient form.

When the orbit is a geodesic, r- and θ-motions are periodic. Then it is convenient to describe a geodesic with phase functions χ_r and χ_θ as

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[††]This time scale is called the dephasing time. [12]

$$R(\mathcal{E},\chi_r) = \sum_n R^{(n)}(\mathcal{E}) e^{i2\pi n\chi_r}\,, \quad \frac{d}{d\lambda}\chi_r \;=\; \frac{1}{\lambda_r(\mathcal{E})}\,, \tag{3.83}$$

$$\Theta(\mathcal{E},\chi_\theta) = \sum_n \Theta^{(n)}(\mathcal{E}) e^{i2\pi n\chi_\theta}\,, \quad \frac{d}{d\lambda}\chi_\theta \;=\; \frac{1}{\lambda_\theta(\mathcal{E})}\,. \tag{3.84}$$

We consider the similar type of description when we consider the self-force. One can rewrite (3.48) and (3.50) as

$$\delta r = \bar{R}_i\delta\mathcal{E}^i + \bar{R}_\chi\delta\chi_r\,, \quad \bar{R}_i \;=\; \sum_n R^{(n)}_{,i} e^{i2\pi n\chi_{r0}}\,, \quad \bar{R}_\chi \;=\; \sum_n i2\pi n R^{(n)} e^{i2\pi n\chi_{r0}}\,, \tag{3.85}$$

$$\delta\theta = \bar{\Theta}_i\delta\mathcal{E}^i + \bar{\Theta}_\chi\delta\chi_\theta\,, \quad \bar{\Theta}_i \;=\; \sum_n \Theta^{(n)}_{,i} e^{i2\pi n\chi_{\theta0}}\,, \quad \bar{\Theta}_\chi \;=\; \sum_n i2\pi n \Theta^{(n)} e^{i2\pi n\chi_{\theta0}}\,, \tag{3.86}$$

where χ_{r0} and $\chi_{\theta0}$ describe the phase functions of the initial geodesic, and $\delta\chi_r$ and $\delta\chi_\theta$ are the deviation by the self-force satisfying

$$\delta\chi_r = -\frac{\lambda-\bar{\lambda}_{r0}}{\lambda_r}\frac{\lambda_{r,i}}{\lambda_r}\delta\mathcal{E}^i - \frac{\delta\bar{\lambda}_{r0}}{\lambda_r}\,, \quad \delta\chi_\theta \;=\; -\frac{\lambda-\bar{\lambda}_{\theta0}}{\lambda_\theta}\frac{\lambda_{\theta,i}}{\lambda_\theta}\delta\mathcal{E}^i - \frac{\delta\bar{\lambda}_{\theta0}}{\lambda_\theta}\,. \tag{3.87}$$

We find that, with the effect of the self-force, the phase functions satisfy

$$\frac{d}{d\lambda}(\chi_r + \delta\chi_{r0}) = \frac{1}{\lambda_r + \lambda_{r,i}\delta\mathcal{E}^i} + F_{\chi_r}(\mathcal{E}_0,\chi_{r0},\chi_{\theta0})\,, \quad F_{\chi_r} \;=\; -\frac{1}{\lambda_r}\sum_{m,n} R_e^{(m,n)} e^{i2\pi(m\chi_{r0}+n\chi_{\theta0})}\,, \tag{3.88}$$

$$\frac{d}{d\lambda}(\chi_\theta + \delta\chi_{\theta0}) = \frac{1}{\lambda_\theta + \lambda_{\theta,i}\delta\mathcal{E}^i} + F_{\chi_\theta}(\mathcal{E}_0,\chi_{r0},\chi_{\theta0})\,, \quad F_{\chi_\theta} \;=\; -\frac{1}{\lambda_r}\sum_{m,n} \Theta_e^{(m,n)} e^{i2\pi(m\chi_{r0}+n\chi_{\theta0})}\,. \tag{3.89}$$

One can see that (3.87) becomes that of a geodesic when we switch off the self-force, thus, by the renormalized perturbation method, we formally obtain the orbital evolution eqution with the self-force as

$$\frac{d}{d\lambda}\mathcal{E} = F_{\mathcal{E}}(\mathcal{E},\chi_r,\chi_\theta)\,, \quad \frac{d}{d\lambda}\chi_r \;=\; \frac{1}{\lambda_r(\mathcal{E})} + F_{\chi_r}(\mathcal{E},\chi_r,\chi_\theta)\,, \quad \frac{d}{d\lambda}\chi_\theta \;=\; \frac{1}{\lambda_\theta(\mathcal{E})} + F_{\chi_\theta}(\mathcal{E},\chi_r,\chi_\theta)\,, \tag{3.90}$$

$$r = \sum_n R^{(n)}(\mathcal{E}) e^{i2\pi n\chi_r}\,, \quad \theta \;=\; \sum_n \Theta^{(n)}(\mathcal{E}) e^{i2\pi n\chi_\theta}\,, \quad \frac{d}{d\lambda}t \;=\; X(\mathcal{E},r,\theta)\,, \quad \frac{d}{d\lambda}\phi \;=\; Y(\mathcal{E},r,\theta)\,, \tag{3.91}$$

where $F_{\mathcal{E}}$ is given by (2.48), (2.49) and (2.50).

The evaluation of (3.90) and (3.91) may be possible by a future investigation of a regularization calculation [11]. But, the calculation may be complex and costly unless we have a new breakthrough in the regularization calculation strategy. We, therefore, introduce an approximate calculation using only (2.61), of which we already have a well-established calculation technique and a number of results [3].

From the perturbative result in Subsec.III A and Subsec.III B, the orbital evolution is dominantely described as

$$\delta\mathcal{E} = \lambda\dot{\mathcal{E}}^{R-ret.(0,0)}(\mathcal{E}_0)\,, \quad \delta\bar{\lambda}_r \;=\; -\frac{\lambda^2}{2}\frac{\lambda_{r,i}}{\lambda_r}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}(\mathcal{E}_0)\,, \quad \delta\bar{\lambda}_\theta \;=\; -\frac{\lambda^2}{2}\frac{\lambda_{\theta,i}}{\lambda_\theta}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}(\mathcal{E}_0)\,, \tag{3.92}$$

$$\delta\bar{t} = -\frac{\lambda^2}{2}\dot{T}_{,i}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}(\mathcal{E}_0)\,, \quad \delta\bar{\phi} \;=\; -\frac{\lambda^2}{2}\dot{\Phi}_{,i}\dot{\mathcal{E}}^{R-ret.\,(0,0)\,i}(\mathcal{E}_0)\,. \tag{3.93}$$

We call the calculation using these dominant parts by an approximate adiabatic calculation, and the evolution equation becomes

$$\frac{d}{d\lambda}\mathcal{E}^{adi.} = \dot{\mathcal{E}}^{R-ret.}(\mathcal{E}^{adi.})\,, \quad \frac{d}{d\lambda}\chi_r^{adi.} \;=\; \frac{1}{\lambda_r(\mathcal{E}^{adi.})}\,, \quad \frac{d}{d\lambda}\chi_\theta^{adi.} \;=\; \frac{1}{\lambda_\theta(\mathcal{E}^{adi.})}\,, \tag{3.94}$$

$$r^{adi.} \;=\; \sum_n R^{(n)}(\mathcal{E}^{adi.}) e^{i2\pi n\chi_r^{adi.}}\,, \quad \theta^{adi.} \;=\; \sum_n \Theta^{(n)}(\mathcal{E}^{adi.}) e^{i2\pi n\chi_\theta^{adi.}}\,, \tag{3.95}$$

$$\frac{d}{d\lambda}t^{adi.} = X(\mathcal{E}^{adi.},r^{adi.},\theta^{adi.})\,, \quad \frac{d}{d\lambda}\phi^{adi.} \;=\; Y(\mathcal{E}^{adi.},r^{adi.},\theta^{adi.})\,. \tag{3.96}$$

For a practical use of this approximate adiabatic calculation, we make a rough estimate on how correctly the orbit can be predicted by this method. We suppose to divide the whole domain of λ into an infinite number of domains $\lambda_k < \lambda < \lambda_{k+1}$ whose interval is $O(\mu^\alpha)$, $\alpha \to -1/2$. Then we may apply our perturbative analysis of the orbital

evolution at each domain. We define the orbital 'constants' at $\lambda = \lambda_k$ as $\{\mathcal{E}^{(k)}, \bar{\lambda}_r^{(k)}, \bar{\lambda}_\theta^{(k)}, \bar{t}^{(k)}, \bar{\phi}^{(k)}\}$. Again, for a definite discussion, we assume that the self-force begins to act at $\lambda = \lambda_0$.

By the result of Subsec.III A and Subsec.III B, a dominant contribution makes the evolution of these 'constants' as

$$\begin{aligned} &\mathcal{E}^{(k+1)} - \mathcal{E}^{(k)} \sim O(\mu^{1+\alpha}), \quad \bar{\lambda}_r^{(k+1)} - \bar{\lambda}_r^{(k)} \sim O(\mu^{1+2\alpha}), \quad \bar{\lambda}_\theta^{(k+1)} - \bar{\lambda}_\theta^{(k)} \sim O(\mu^{1+2\alpha}), \\ &\bar{t}^{(k+1)} - \bar{t}^{(k)} \sim O(\mu^{1+2\alpha}), \quad \bar{\phi}^{(k+1)} - \bar{\phi}^{(k)} \sim O(\mu^{1+2\alpha}). \end{aligned} \tag{3.97}$$

After passing by $N \sim O(\mu^\beta)$ finite domains, we have

$$\begin{aligned} &\mathcal{E}^{(N)} \sim O(\mu^{1+\alpha+\beta}), \quad \bar{\lambda}_r^{(N)} \sim O(\mu^{1+2\alpha+\beta}), \quad \bar{\lambda}_\theta^{(N)} \sim O(\mu^{1+2\alpha+\beta}), \\ &\bar{t}^{(N)} \sim O(\mu^{1+2\alpha+\beta}), \quad \bar{\phi}^{(N)} \sim O(\mu^{1+2\alpha+\beta}). \end{aligned} \tag{3.98}$$

On the other hand, the part we ignore for the approximate adiabatic calculation contaminates the evolution as

$$\begin{aligned} &\delta(\mathcal{E}^{(k+1)} - \mathcal{E}^{(k)}) \sim O(\mu), \quad \delta(\bar{\lambda}_r^{(k+1)} - \bar{\lambda}_r^{(k)}) \sim O(\mu^{1+\alpha}), \quad \delta(\bar{\lambda}_\theta^{(k+1)} - \bar{\lambda}_\theta^{(k)}) \sim O(\mu^{1+\alpha}), \\ &\delta(\bar{t}^{(k+1)} - \bar{t}^{(k)}) \sim O(\mu^{1+\alpha}), \quad \delta(\bar{\phi}^{(k+1)} - \bar{\phi}^{(k)}) \sim O(\mu^{1+\alpha}). \end{aligned} \tag{3.99}$$

We assume that the ignored part affects the evolution like an randam Gaussian noise and estmate the error as

$$\begin{aligned} &\delta\mathcal{E}^{(N)} \sim O(\mu^{1+\beta/2}), \quad \delta\bar{\lambda}_r^{(N)} \sim O(\mu^{1+\alpha+\beta/2}), \quad \delta\bar{\lambda}_\theta^{(N)} \sim O(\mu^{1+\alpha+\beta/2}), \\ &\delta\bar{t}^{(N)} \sim O(\mu^{1+\alpha+\beta/2}), \quad \delta\bar{\phi}^{(N)} \sim O(\mu^{1+\alpha+\beta/2}). \end{aligned} \tag{3.100}$$

Using this estimation, we discuss the predictability of the approximate adiabatic calculate. The error of the rotation per time ϕ/t is estimated as

$$\delta\left(\frac{\phi}{t}\right) \sim O(\mu^{-\alpha-\beta/2}). \tag{3.101}$$

For an accurate prediction, we require $\delta(\phi/t) < 1$ and we have $\beta < -2\alpha \to 1$, which is always satisfied since N is just an integer.

The error of the rotation phase ϕ is estimated as

$$\delta\phi \sim O(\mu^{1+\alpha+\beta/2}). \tag{3.102}$$

For a correct prediction of the phase, we have $\beta > -2\alpha - 2 \to -1$. This shows that we have a prediction of the rotation phase only at $\lambda \leq O(\mu^{-3/2})$.

D. Gauge Issue

The gravitational self-force problem has an exceptional difficulty because of the so-called gauge problem [13]. The regularization formulation was originally formulated in the harmonic gauge condition [4,7], and we have the divergent S-part only in the harmonic gauge. It was pointed out that, when we subtract the S-part from the full metric perturbation in the radiation gauge [10], we have a divergent residue [13] because of the divergent gauge transformation at the particle location. For this reason, the calculation of the full metric perturbation in the harmonic gauge condition, or the calculation of the S-part of the metric perturbation in the radiation gauge become important issues in calculating the gravitational self-force in a Kerr background.

On the other hand, in the approximate adiabatic calculation, all we need is to evaluate (2.61), which is proven to agree with the radiation reaction calculation [8] in part. A number of previous works [3] prove there is no divergence in the approximate adiabatic calculation. The investigation in Subsec.II D shows that the self-force derived by the radiative Green function corresponds to the two-point averaged self-force derived by the R-part of the retarded Green function. We consider that, by taking a two point average of the self-force, the divergent S-part vanishes as in (2.38). We consider that, in the approximate adiabatic calculation, the same cancellation mechanism happens including divergence by the gauge transformation. But there still be an abiguity of a finite gauge choice. Here we prove the result by the approximate adiabatic calculation is actually gauge-invariant by showing the gauge invariance of (2.61).

Using a Killing vector η_α and a Killing-Yano tensor $\eta_{\alpha\beta}$ of a Kerr spacetime, conserved quantities along a geodesic are described as

$$E = \eta_\alpha v^\alpha, \quad Q = \eta_{\alpha\beta} v^\alpha v^\beta, \tag{3.103}$$

where $v^\alpha(\tau)=dz^\alpha/d\tau$ is 4-velocity and τ is proper time of the geodesic. (2.61) can be rewritten as

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\tau\frac{d}{d\tau}E=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\tau\eta_\alpha(z(\tau))F^\alpha(\tau)\,, \tag{3.104}$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\tau\frac{d}{d\tau}Q=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\tau 2\eta_{\alpha\beta}(z(\tau))v^\alpha(\tau)F^\beta(\tau)\,, \tag{3.105}$$

where F^β is the self-force vector.

By a gauge transformation $x^\alpha\to x^\alpha+\xi^\alpha$, the self-force is transformed as

$$F^\alpha(\tau)\to F^\alpha(\tau)+\delta_\xi F^\alpha(\tau)\,, \tag{3.106}$$

$$\delta_\xi F^\alpha(\tau)\,=\,-\Big(v^\beta(\tau)v^\gamma(\tau)\xi^\alpha{}_{;\beta\gamma}(z(\tau))+R^\alpha{}_{\beta\gamma\delta}(z(\tau))v^\beta(\tau)\xi^\gamma(z(\tau))v^\delta(\tau)\Big)\,. \tag{3.107}$$

The increment of (3.104) and (3.105) by this extra term $\delta_\xi F^\alpha(\tau)$ becomes

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\tau\delta\left[\frac{d}{d\tau}E(\tau)\right]\;=\lim_{T\to\infty}\frac{1}{2T}\left[-\eta^\alpha v^\beta\xi_{\alpha;\beta}+v^\beta\eta^\alpha{}_{;\beta}\xi_\alpha\right]_{-T}^{T}\,, \tag{3.108}$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\tau\delta\left[\frac{d}{d\tau}Q(\tau)\right]\;=\lim_{T\to\infty}\frac{1}{2T}\left[-2v_\beta\eta^{\alpha\beta}v^\gamma\xi_{\alpha;\gamma}+2v_\beta v^\gamma\eta^{\alpha\beta}{}_{;\gamma}\xi_\alpha\right]_{-T}^{T}\,, \tag{3.109}$$

where we use Killing equations $\eta_{\alpha;\beta\gamma}=\eta_\delta R^\delta{}_{\gamma\beta\alpha}$ and $\eta_{\alpha\beta;\gamma\delta}=\eta_{\epsilon\beta}R^\epsilon{}_{\delta\gamma\alpha}+\eta_{\alpha\epsilon}R^\epsilon{}_{\delta\gamma\beta}$. Since the gauge dependence is totally integrated out, the gauge dependence of the approximate adiabatic calculation vanishes by taking $T\to\infty$.

We comment that the gauge dependence of (3.90) and (3.91) is highly non-trivial and we left it as a future problem.

IV. SUMMARY

In this paper, we discuss a method to calculate an orbital evolution by a scalar/electromagnetic/gravitational self-force. We assume that the self-force is weak and that the orbit can be approximated by a geodesic at each instant of time, with which one can derive the self-force. We note that the geodesic equation is a set of second-order differential equations of four components, and we have 7 integral constants, E, L, K, λ_0, λ_1, t_0 and ϕ_0. Instead of calculating the orbit itself, we derive the equations of these 'constants' by the self-force under this assumption.

We first consider the evolution of E, L and K since the evolution equations of these 'constants' are directly derived by the self-force vector. We exploit the symmetry of a Kerr spacetime together with a family of geodesics, which induce the self-force. By applying the symmetry transformation to the self-force vector, we find that the time-averaged evolution of E, L and K can be derived by using a radiative Green function, which has a number of technical advantage in practice. However, we also find that the orbit does not evolve adiabatically in an exact sense.

In order to understand the orbital evolution by the self-force, we next consider the orbital equation in a perturbative manner. We integrate the orbital equation by a time scale, sufficiently long, but, less than the dephasing time when the linear perturbation of the orbit becomes invalid. Since the orbit does not evolve in an adiabatic manner, the orbital constants of a geodesic $\{\mathcal{E},\bar{\lambda}_r,\bar{\lambda}_\theta,\bar{t},\bar{\phi}_0\}$ are oscillating by the self-force. However, we could find out secularly growing parts which will dominate the orbital evolution. By taking these growing parts only, we define an approximate orbital eequation, which we call an approximate adiabatic calculation. We consider that the approximate adiabatic calculation is enough implementable by a well-established method since it only uses the radiative Green function.

We also discuss how approximate an orbital evolution can be obtained by this calculation method. We find that, during the time $O(\mu^{-3/2})$, it gives an accurate rotation phase. For example, when 10 solar-mass black hole is inspiralling into 10^7 solar-mass supermassive black hole, this corresponds to around 10^9-rotation period. Though the accuracy in predicting a wave form is not clear in our estimate, if only the accurate prediction of the rotation phase is important for the future LISA observation, the approximation proposed here may give a sufficient imformation in this case.

Finally we prove that the approximate adiabatic calculation gives a gauge invariant prediction, thus, the result is consistent with that by another possible method within the approximation scheme.

ACKNOWLEDGEMENTS

We thank Professor Clifford Will for encouragement. Special thanks to Professor Misao Sasaki and Professor Takahiro Tanaka for encouragement and a fruitful discussion with a hospitality during the stay in Osaka University and Kyoto University. We also thank Dr. Jhingan Sanjay and Dr. Hiroyuki Nakano for a discussion. This work was supported in part by the National Science Foundation under grant number PHY 00-96522.

APPENDIX: CIRCULAR OR EQUATORIAL ORBIT

When the orbit is circular, or equatorial, we have $dr_0/d\lambda = V(\mathcal{E}_0, r_0) = V(\mathcal{E}_0, r_0)_{,r} = 0$, or $d\theta_0/d\lambda = U(\mathcal{E}_0, \theta_0) = U(\mathcal{E}_0, \theta_0)_{,\theta} = 0$. Then (3.8) becomes trivial and we cannot evaluate the perturbed orbital equation. Here we consider the orbital evolution in these special cases by taking the circular, or equatorial limit of a general orbit. Before the discussion, we note that, by repeating the symmetry argument in Sec.II, the self-force acting on E, L and K becomes

$$\left[\frac{d}{d\lambda}\mathcal{E}\right]^{R-ret./R-adv./rad.} = \begin{cases} \sum_n \dot{\mathcal{E}}^{R-ret./R-adv./rad.(0,n)}(E,L,K)e^{i2\pi n\frac{\lambda-\bar{\lambda}_\theta}{\lambda_\theta}}\,, & \text{circular orbit}\ , \\ \sum_m \dot{\mathcal{E}}^{R-ret./R-adv./rad.(m,0)}(E,L,K)e^{i2\pi m\frac{\lambda-\bar{\lambda}_r}{\lambda_r}}\,, & \text{equatorial orbit}\ , \\ \dot{\mathcal{E}}^{R-ret./R-adv./rad.(0,0)}(E,L,K)\,, & \text{circular and equatorial orbit}\ . \end{cases} \tag{A1}$$

We consider r-motion when we consider a self-force acting on a circular orbit. The perturbation equation (3.8) becomes trivial since we have $V_{,i} = V_{,r} = 0$ along the orbit by using (3.12). Instead of calculating the orbital equation, we consider to take the circular limit of (3.48). The first term of (3.48) is given in (3.27) and behaves in a regular manner when we take a circular limit. On the other hand, the second term of (3.48) is given in (3.44). By the regularization calculation starting from (3.32), the integration is still finite in the circular limit, and $dr_0/d\lambda$ vanishes. Thus, we have

$$\delta r = R_{,i}\delta\mathcal{E}^i\,, \tag{A2}$$

which means the circular orbit stays circular under the self-force and the orbit is solely determined by E, L and K [‡‡] As a result, the orbital equation becomes

$$\frac{d}{d\lambda}\mathcal{E} = F_{\mathcal{E}}(\mathcal{E},\chi_\theta)\,, \quad \frac{d}{d\lambda}\chi_\theta \,=\, \frac{1}{\lambda_\theta(\mathcal{E})} + F_{\chi_\theta}(\mathcal{E},\chi_r,\chi_\theta)\,, \tag{A3}$$

$$r = R^{(0)}(\mathcal{E})\,, \quad \theta \,=\, \sum_n \Theta^{(n)}(\mathcal{E})e^{i2\pi n\chi_\theta}\,, \quad \frac{d}{d\lambda}t \,=\, X(\mathcal{E},r,\theta)\,, \quad \frac{d}{d\lambda}\phi \,=\, Y(\mathcal{E},r,\theta)\,, \tag{A4}$$

and, under the approximate adiabatic calculation, we have

$$\frac{d}{d\lambda}\mathcal{E}^{adi.} = \dot{\mathcal{E}}^{R-ret.}(\mathcal{E}^{adi.})\,, \quad \frac{d}{d\lambda}\chi_\theta^{adi.} \,=\, \frac{1}{\lambda_\theta(\mathcal{E}^{adi.})}\,, \tag{A5}$$

$$r^{adi.} = R^{(0)}(\mathcal{E}^{adi.})\,, \quad \theta^{adi.} \,=\, \sum_n \Theta^{(n)}(\mathcal{E}^{adi.})e^{i2\pi n\chi_\theta^{adi.}}\,, \tag{A6}$$

$$\frac{d}{d\lambda}t^{adi.} = X(\mathcal{E}^{adi.},r^{adi.},\theta^{adi.})\,, \quad \frac{d}{d\lambda}\phi^{adi.} \,=\, Y(\mathcal{E}^{adi.},r^{adi.},\theta^{adi.})\,. \tag{A7}$$

Similarly, θ-motion can be drived such that the equatorial motion stays equatorial. One can also prove by using the symmetry $\theta \to -\theta$. Using this symmetry property, when the orbit is equatorial, the self-force satisfies $F_\theta = 0$, and we have $\theta = \pi/2$, $d\theta/d\lambda = 0$. The orbital equation becomes

$$\frac{d}{d\lambda}\mathcal{E} = F_{\mathcal{E}}(\mathcal{E},\chi_r)\,, \quad \frac{d}{d\lambda}\chi_r \,=\, \frac{1}{\lambda_\theta(\mathcal{E})} + F_{\chi_\theta}(\mathcal{E},\chi_\theta)\,, \tag{A8}$$

$$r = \sum_n R^{(n)}(\mathcal{E})e^{i2\pi n\chi_r}\,, \quad \theta \,=\, \frac{\pi}{2}\,, \quad \frac{d}{d\lambda}t \,=\, X(\mathcal{E},r,\pi/2)\,, \quad \frac{d}{d\lambda}\phi \,=\, Y(\mathcal{E},r,\pi/2)\,, \tag{A9}$$

and, under the approximate adiabatic calculation, we have

$$\frac{d}{d\lambda}\mathcal{E}^{adi.} = \dot{\mathcal{E}}^{R-ret.}(\mathcal{E}^{adi.})\,, \quad \frac{d}{d\lambda}\chi_r^{adi.} \,=\, \frac{1}{\lambda_r(\mathcal{E}^{adi.})}\,, \tag{A10}$$

$$r^{adi.} = \sum_n R^{(n)}(\mathcal{E}^{adi.})e^{i2\pi n\chi_r^{adi.}}\,, \quad \theta^{adi.} \,=\, \pi/2\,, \tag{A11}$$

$$\frac{d}{d\lambda}t^{adi.} = X(\mathcal{E}^{adi.},r^{adi.},\pi/2)\,, \quad \frac{d}{d\lambda}\phi^{adi.} \,=\, Y(\mathcal{E}^{adi.},r^{adi.},\pi/2)\,. \tag{A12}$$

[‡‡]The same proof was given in Ref. [14].

When the orbit is circular and equatorial, the oribtal equation becomes

$$\frac{d}{d\lambda}\mathcal{E} = F_{\mathcal{E}}(\mathcal{E})\,, \tag{A13}$$

$$r = R^{(0)}(\mathcal{E})\,, \quad \theta = \frac{\pi}{2}\,, \quad \frac{d}{d\lambda}t = X(\mathcal{E}, r, \pi/2)\,, \quad \frac{d}{d\lambda}\phi = Y(\mathcal{E}, r, \pi/2)\,, \tag{A14}$$

and the orbit evolves adiabatically in an exact sense.

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